

# The Geometry of Oriented Cubes

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## Abstract

This reports on the fundamental objects revealed by Ross Street, which he called ‘orientals’. Street’s work was in part inspired by Robert’s attempts [8] to use  $n$ -category ideas to construct nets of  $C^*$ -algebras in Minkowski space for applications to relativistic quantum field theory: Roberts’ additional challenge was that “no amount of staring at the low dimensional cocycle conditions would reveal the pattern for higher dimensions”.

This report takes up this challenge, presents a natural inductive construction of explicit cubical cocycle conditions, and gives three ways in which the simplicial ones can be derived from these. A consequence of this work is that the Yang-Baxter equation, the ‘pentagon of pentagons’, and higher simplex equations, are in essence different manifestations of the same underlying abstract structure.

There has been recent interest in higher-categories, by computer scientists investigating concurrency theory, as well as by physicists, among others. The dual ‘string’ version of this paper, originally presented in [2], and which motivated much of the role of cubes in [10], makes clear the relationship with higher-dimensional simplex equations in physics.

Much work in this area has been done since these notes were written: no attempt has been made to update the original report. However, all diagrams have been redrawn by computer, replacing all original hand-drawn pictures.

We obtain a geometric interpretation of Street’s simplicial orientals, by describing an oriental structure on the  $n$ -cube. This corresponds to describing nested sequences of embedded disks in the boundary of the  $n$ -cube, obtained consecutively as ordered deformations keeping the boundaries fixed. Canonical forms for the order are obtained, leading to definitions of cocycle conditions for application to non-abelian cohomology in a category.

# 1 Introduction

Street [9] has defined an  $n$ -category structure on the  $k$ -simplex, leading to his notion of *orientals*. This was motivated by an approach to non-abelian cohomology, arising from quantum field theory [8], but also seems to have intriguing connections with homotopy coherence, loop space desuspension and the cobar construction, as well as the realisation of homotopy types. Intimately related to Street’s structure on simplices is an analogous one on cubes, from which the simplicial results can be derived. This paper takes up Roberts’ challenge [8] that “no amount of staring at the low dimensional cocycle conditions would reveal the pattern for higher dimensions”. The result is surprisingly simple, natural and beautiful. We define and describe this finer structure of oriented cubes, from a geometric point of view. This makes obvious the relevance to homotopy theory.

This paper is self-contained, presupposing minimal familiarity with either category theory or homotopy theory. Categorical aspects and applications will be discussed in a subsequent paper.

Consider the following geometric reasoning: In obstruction theory applied to simplicial or cubical complexes, a standard problem is the continuous extension of data defined on the boundary of a cell (usually a simplex or cube). Although the boundary of such a cell is a geometrically decomposed sphere, the only structure used involves assigning alternating signs appropriately to the sub-cells of the decomposition. If we wish to keep the spherical structure in mind, we can imagine the boundary sphere of the cell decomposed as two hemispheres, with an extension over the cell of the data assigned to the hemispheres now viewed as a continuous deformation, keeping the boundary data fixed, between the assignments of data to each of the hemispheres. Since the boundary of each hemisphere is also a sphere, it is natural to seek splittings of spheres in every dimension.

Suppose we have some “data” defined over a sphere, and two possibly distinct extensions over a cell. By viewing two cells as upper and lower hemispheres, we may instead consider this as an extension, over a sphere of one dimension higher, of data assigned to the “equatorial” sphere. We can then think of the two extensions over the hemispheres as being “equivalent” in some sense if the data assigned to this higher sphere extends over a cell. But there may be another such extension, tantamount to an alternative equivalence, which enables us to bootstrap the procedure. At any stage, we see data on a sphere, split as two hemispheres, with data on the equatorial sphere, which in turn splits as two hemispheres, whose boundary sphere splits as two hemispheres, and so on.

Consider now category theory, viewing arrows (or morphisms) as one dimensional cells whose boundary 0-spheres split as the union of a source and a target - which we view as 0-dimensional points. If we consider a 2-category (see [6] for example), we see that 2-cells have a 1-dimensional source and target, each of which is an ordinary arrow in a category, and both of which have the same 0-dimensional source and target. Thus we see the boundary of a 2-cell split into two hemispheres, and the boundary of each hemisphere similarly split. (Figure 1)

Street’s *oriental* structure on simplices [9] allows an interpretation as a coherent version, across all dimensions, of a structure as described above. Our purpose is to exhibit such a structure on the  $n$ -dimensional cube. This leads to the notion of a “ $k$ -dimensional source” and “ $k$ -dimensional target” of the  $n$ -cube, for each  $k$ , just as a 2-cell in a 2-category has sources in dimensions 0 and 1. Amusingly, the answer suggests a

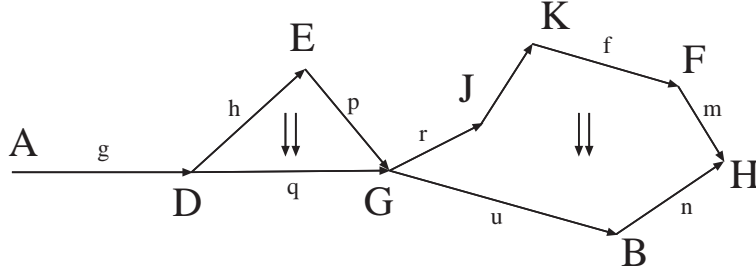


Figure 1: Typical 2-category pasting scheme: boundaries of 2-cells naturally split as source- and target-hemispheres

curious connection with  $p$ -forms on  $\mathbb{R}^n$ . From a categorical viewpoint this decomposition leads to the notion of an  $n$ -category.

## 2 The structure of the $n$ -cube

Denote by  $\underline{n}$  the (ordered) set  $\{1, 2, \dots, n\}$ , and by  $\mathcal{I} = \{-, 0, +\} \cong \mathbb{Z}_3$ . Let  $I$  denote the interval  $[-1, 1]$  in the real number axis, and  $I^n$  the (geometric)  $n$ -dimensional cube. We think of a point as the 0-dimensional cube or simplex.

**Dimension 1:** In Figure 2 we have a single  $I^1$ , with a notion of source (“0-source”) and target. We may think of this as an arrow assigned to the interval  $I$ . As a convention, we choose “ $-1$ ” to be considered the source and “ $+1$ ” the target. Hence the dichotomy “source/target” arises from the disconnectedness of the 0-sphere  $S^0 = \partial I$ . It is convenient for “ $-$ ” (respectively “ $+$ ”) to be considered synonymous with “ $-1$ ” (respectively  $+1 = 1$ ). Moreover, we use “ $0$ ” to denote the whole interval:

Observe that  $I^1$  has three sub-cubes,  $+$ ,  $0$  and  $-$ . When we take the product with the unit interval we obtain  $I^2$ , to which each sub-cube of  $I^1$  contributes three sub-cubes. These can be thought of as a new copy, an old copy and a thickened copy. Proceeding inductively we see that  $I^n$  has  $3^n$  sub-cubes. More precisely, let  $\mathcal{I}^n = \{x : \underline{n} \longrightarrow \mathcal{I}\}$ .

**Proposition 2.1:** There is a 1:1 correspondence between (geometric) sub-cubes of  $I^n$  and elements of  $\mathcal{I}^n$ .

Thus we may *define* the  $n$ -cube to be  $\mathcal{I}^n$ , and call an element  $x$  of  $\mathcal{I}^n$  a *sub-cube* of  $\mathcal{I}^n$ . We will henceforth use the notation  $\mathcal{I}^n$ , or occasionally  $[n]$ , to denote the  $n$ -cube, and no longer use  $I^n$ .

**Notation:** Since each  $\underline{n}$  is ordered, an element  $x$  of  $\mathcal{I}^n$  can be specified as a word of length  $n$  in the symbols of  $\mathcal{I}$ .

**Example 2.2:** In Figure 2 we draw the 1, 2, and 3 dimensional cubes, labeled according to the usual convention arising from the unit interval in  $\mathbb{R}^1$ .

Define maps  $\lambda, \mu, \nu : \mathcal{I}^n \longrightarrow \mathcal{I}^{n+1}$  by

$$\begin{aligned} (\lambda(x))(i) &= x(i) \quad \text{for } i \leq n, & (\lambda(x))(n+1) &= - \\ (\mu(x))(i) &= x(i) \quad \text{for } i \leq n, & (\mu(x))(n+1) &= 0 \\ (\nu(x))(i) &= x(i) \quad \text{for } i \leq n, & (\nu(x))(n+1) &= + \end{aligned}$$

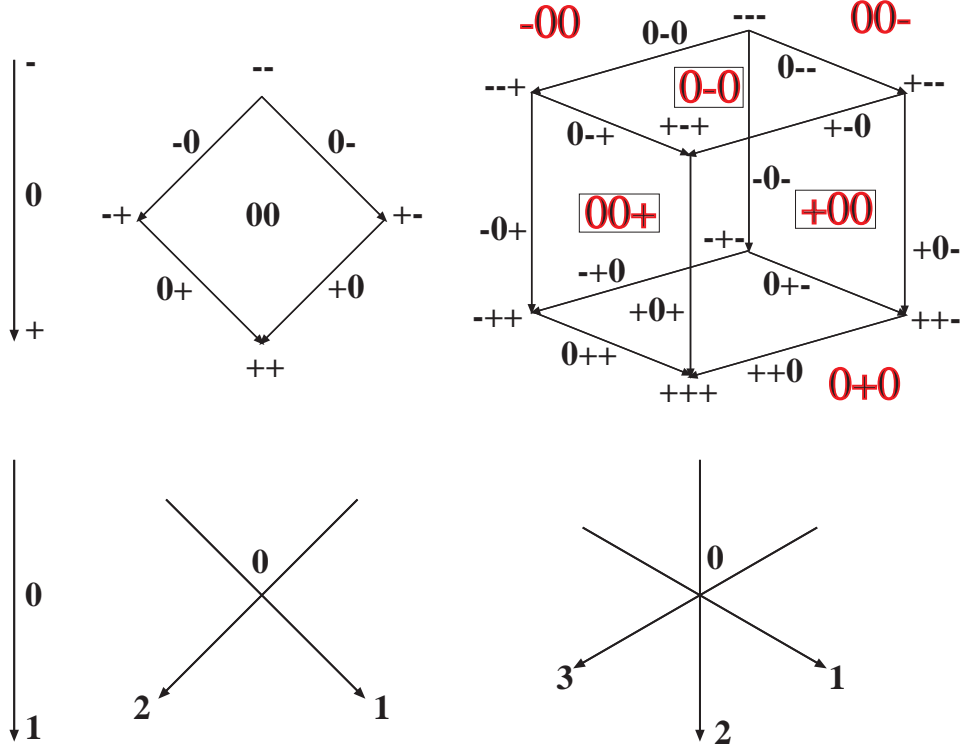


Figure 2: Labeling of cubes in dimensions 1,2,3, and their dual representations

These add to the right of a word the symbols  $-$ ,  $0$  and  $+$  respectively, and can be thought of as the old, thickened and new copies of  $x$ .

**Definition 2.3:** The *dimension* of  $x$ , denoted by  $|x|$ , is the cardinality of  $x^{-1}(0)$ .

This is the number of  $0$ 's occurring in the word-form of  $x$ , and induces a grading  $\mathcal{I}^n = \sum_{j=0}^n \mathcal{I}_j^n$  by setting  $x \in \mathcal{I}_j^n \iff |x| = j$ .

There are three distinguished subcubes of  $\mathcal{I}^n$ , which in word form are  $0 \cdots 0$ ,  $- \cdots -$  and  $+ \cdots +$ . We shall respectively refer to the latter two as the  $0$ -source and  $0$ -target of  $\mathcal{I}^n$ , denoted by  $\sigma_0[n]$  and  $\tau_0[n]$ . More generally we will define  $\sigma_k[x]$  and  $\tau_k[x]$  for each  $x \in \mathcal{I}^n$ . These are to be thought of as “ $k$ -dimensional source” and “ $k$ -dimensional target” of  $x$  respectively. We shall suppress some symbols when the context is clear.

For each  $x \in \mathcal{I}^n$  of dimension  $p$  and  $y \in \mathcal{I}^p$ , we can define a new element  $x * y$ . Set  $(x * y)(k) = x(k)$  if  $x(k) \neq 0$ . If  $x^{-1}(0) = \{k_1, \dots, k_p\}$ , listed in increasing order, define  $x * y$  by performing  $x$  on  $x^{-1}(-, +)$  and  $y$  otherwise, using the natural order:  $(x * y)(k_j) = y(j)$ . This gives a pairing

$$\mathcal{I}_p^n \times \mathcal{I}_q^p \longrightarrow \mathcal{I}_q^n.$$

Note that this defines an associative operation as with matrix multiplication.

We may think of any such  $x$  as a  $p$ -cube placed in  $\mathcal{I}^n$ . More generally we can use  $*$  to place any set of sub-cubes of the  $p$ -cube into  $\mathcal{I}^n$ . Once we define  $\sigma_k$  and  $\tau_k$  on  $\mathcal{I}^p$ , we can extend to  $x$  by  $\sigma_k[x] = x * (\sigma_k[p])$ , and similarly for  $\tau_k$ .

### 3 Motivational examples

We examine the lowest dimensional cubes to elucidate their structure:

**Dimension 2:** The boundary of  $\mathcal{I}^2 \cong I^2$  consists of four edges and four vertices. To each of the edges is assigned naturally, given our convention, an orientation. This canonically determines the source and target in dimension 0, as shown in Figure 3. Since the edge  $\{-1\} \times I$  (respectively  $\{1\} \times I$ ) arises as the thickened 0-source (respectively 0-target) of  $\mathcal{I}^1$ , a natural criterion for choosing the one dimensional source and target for  $\mathcal{I}^2$  is determined. Hence we may assign a “double arrow” to  $\mathcal{I}^2$ , pointing from the 1-source to the 1-target. Note that geometrically the boundary circle (1-sphere) of  $\mathcal{I}^2$  is decomposed into two hemispheres, each having as boundary the 0-source and 0-target.

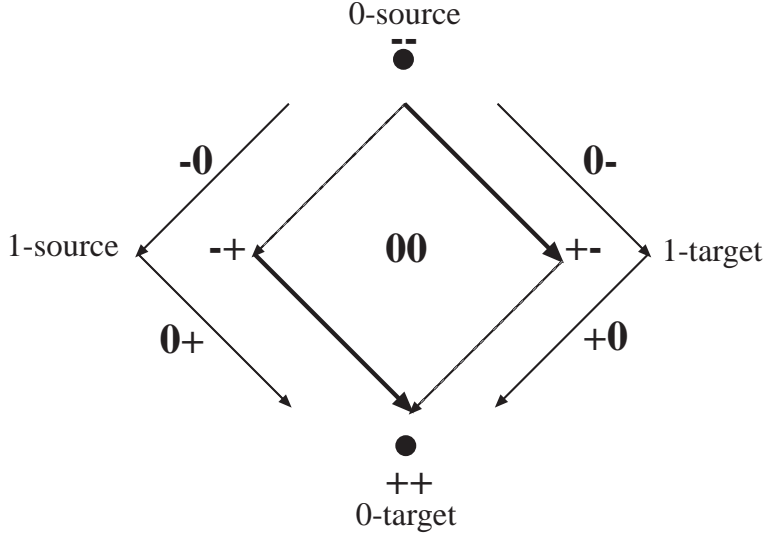


Figure 3: The 2-cube with all sources and targets, which are 00 for dimension 2 or more.

**Dimension 3:** In Figure 4, we thicken the various subcubes of  $\mathcal{I}^2$  to obtain the usual description of  $\mathcal{I}^3$ . Again the 0-dimensional sources and targets are determined, as are the 1-dimensional ones. Observe that the 1-source and 1-target are symmetrically defined, and that each consists of a copy of the 2-dimensional ones, suitably shifted by the third dimension, together with a thickened 0-dimensional source or target. Applying the same reasoning to the definition of 2-source and 2-target as we did for dimension 1, we see the 2-source and 2-target of  $\mathcal{I}^3$  as depicted in Figure 5. The boundary of  $\mathcal{I}^3$  is a 2-sphere splitting as two hemispheres, the 2-dimensional source (at the back) and target (at front) of  $\mathcal{I}^3$ , which in turn have boundary 1-sphere splitting as two hemispheres, the 1-dimensional source and target of  $\mathcal{I}^3$ , which in turn have boundary 0-sphere splitting as two hemispheres, the 0-dimensional source and target of  $\mathcal{I}^3$ .

**Remark:** Although there is obvious appeal in the simplicity of the preceding description of the 2-source and 2-target of  $\mathcal{I}^3$  as the union of three  $\mathcal{I}^2$ s each, observe that the  $\mathcal{I}^2$ s concerned do not have the same 0-source and 0-target, nor the same 1-source and 1-target. Hence a little care is required to get the definitions correct. At this level, the answer both algebraically and geometrically is provided by the notions of 2-category theory and the middle-interchange law ([6, 9]). This is described in Figure 6. Each of the  $\mathcal{I}^2$ s comprising the 2-source of  $\mathcal{I}^3$  must be stretched out by an appropriate  $\mathcal{I}^1$  to the 0-source or 0-target of  $\mathcal{I}^3$ . An interpretation of this goes as follows: each 1-dimensional path from the 0-source to the 0-target can be thought of as the trace of a deformation or homotopy of the 0-source to the 0-target, consecutively across the arrows forming the path.

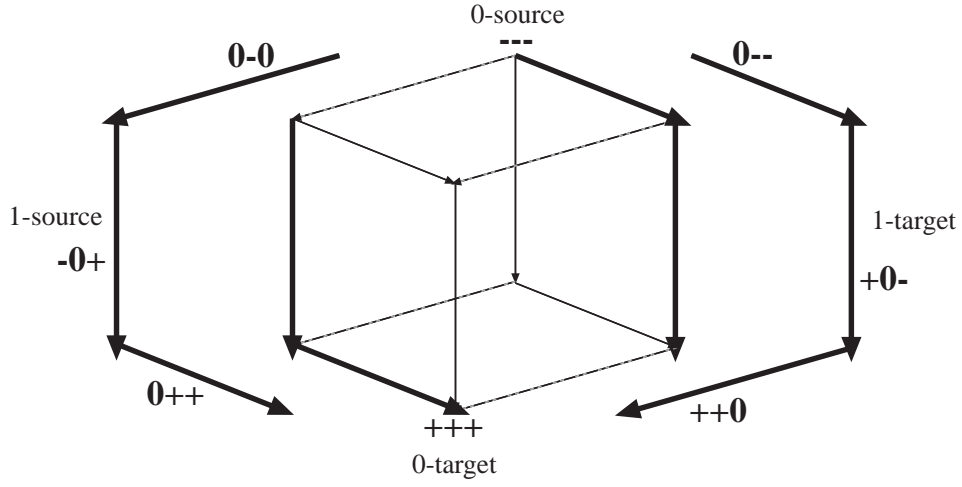


Figure 4: 0- and 1-dimensional source and target of the 3-cube

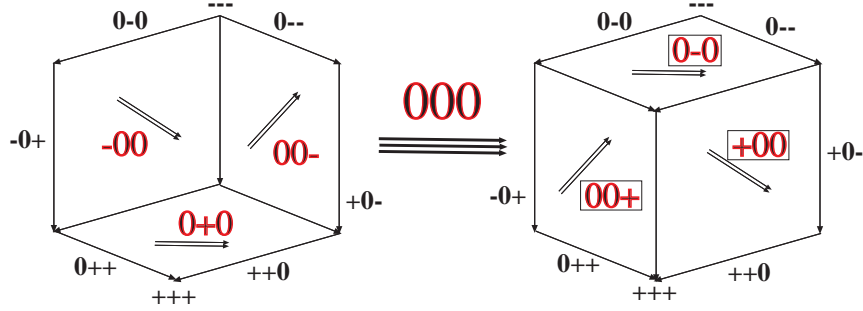


Figure 5: The 2-source and 2-target of the 3-cube

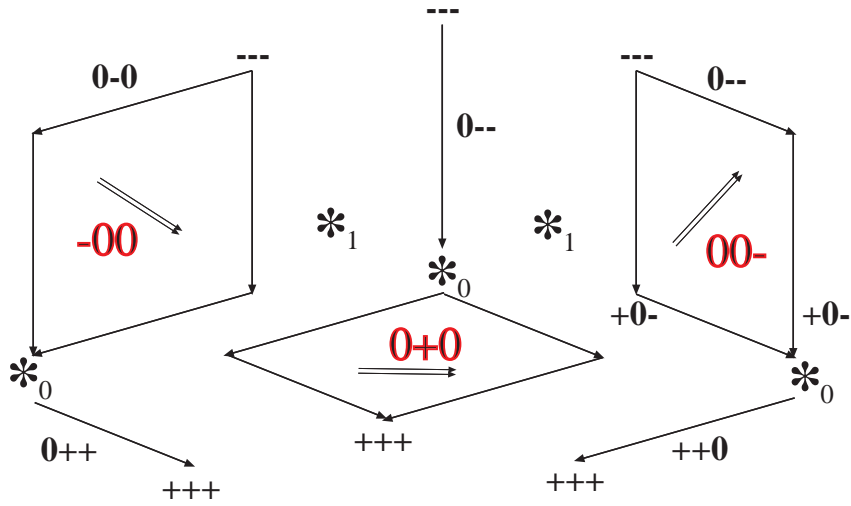


Figure 6: The 2-source of the 3-cube: we add 1-cells to the 2-cells, enabling matching

See Figure 7. In  $\mathcal{I}^2$ , there are only two possible paths, the 1-source and the 1-target, and  $\mathcal{I}^2$  itself can be thought of as providing a deformation from the 1-source to the 1-target. For  $\mathcal{I}^3$ , there are four possible paths for each of the 2-source and 2-target.

In the same way, each of the 2-source and 2-target can be thought of as deformations, in three stages each, of the 1-source to the 1-target. The source deformations must be carried out in the order shown in Figure 8. This is because we cannot deform a 1-path across a  $\mathcal{I}^2$  unless all of the 1-source of the  $\mathcal{I}^2$  in question is part of the 1-path. Each such deformation leaves unchanged the rest of the 1-path, and the number of segments in the path remains the same. Each component of the 2-source consists of a  $\mathcal{I}^2$  across which part of the 1-path is being deformed, and the remainder of the 1-path. From another viewpoint, the  $\mathcal{I}^1$ s string

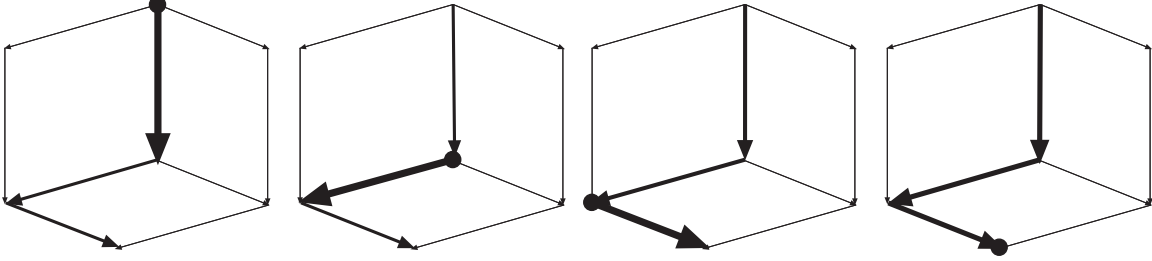


Figure 7: Each 1-path provides a route from the 0-source to the 0-target

out the  $\mathcal{I}^2$ s, providing the path by which the 0-source of  $\mathcal{I}^3$  can be deformed to the 0-source of the relevant  $\mathcal{I}^2$ . This gives rise to the  $*_0$  composition of 2-category theory.

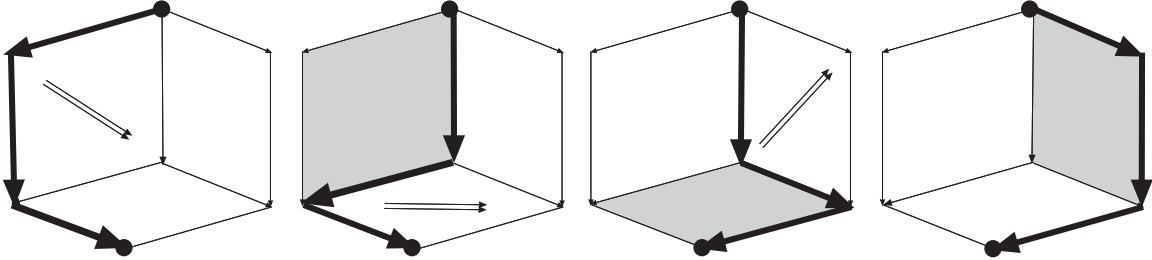


Figure 8: 2-cubes provide deformations of the 1-source to the 1-target; each such deformation first requires the presence of all of its 1-source cubes. White dots represent 0-sources of 2-cubes; grey dots the target 0-cube of the previous 2-cube, which subsequently are moved by 1-cubes.

Note that in this case we are composing cubes stuck end-to-end, source-to-target, along the 0-dimensional (geometric) sources and targets. The  $\mathcal{I}^2$ s, on the other hand, are glued together along 1-dimensional source/target paths. This corresponds to  $*_1$  in the language of 2-categories. The geometry is exactly that of 2-categorical pasting diagrams. (See for example [6].)

Thus not only are the constituent parts of the 2-source of  $\mathcal{I}^3$  derived from the structure of  $\mathcal{I}^2$ , but a natural order emerges for their 2-categorical order of composition. This order in turn derives entirely from our initial convention for source and target of  $\mathcal{I}^1$ . We also note that the 2-cubes of the source or target fit together in such a way that the deformations are possible, and compatibly with the structure defined on  $\mathcal{I}^2$ .

## 4 Source and target faces - Piles

We require  $\mathcal{I}^n$  to have sources and targets in all dimensions  $\leq n$ . These allow us to deform the  $(k-1)$ -source, across either the  $k$ -dimensional source or target, onto the  $(k-1)$ -target. The  $(k-1)$ -source is made up of  $\mathcal{I}^{k-1}$ s

as well as lower dimensional cubes. The lower dimensional cubes stretch out each  $\mathcal{I}^{k-1}$  to the sources and targets of  $\mathcal{I}^n$  in all dimensions  $\leq (k-3)$ .

In deforming the  $(k-1)$ -source across to the  $(k-1)$ -target, such lower dimensional cubes play a role; we will not be able to push across a given  $\mathcal{I}^j$  unless all of *its* lower dimensional sources have been reached. The order in which we proceed is determined inductively, necessitated by the need to expose all lower dimensional sources as required (compare Figure 8). The notion of “pile” enables us to describe the procedure to any desired degree of refinement. A pile is a list of all  $\mathcal{I}^k$ s we have reached at any stage of the deformation. Thus there are source and target piles, and intermediate piles. Each time we deform across a  $\mathcal{I}^k$  we alter the pile in all dimensions less than  $k$ , since we deform from the source of that  $\mathcal{I}^k$  to its target in *all* lower dimensions. In this way sub-cubes act as “operators” on piles. The order of deformation is then determined by groupings of sub-cubes as “blocks”. These blocks are characterised in that the initial and final piles on which they act respectively contain the  $j$ -dimensional source and target faces of the  $\mathcal{I}^n$  in question. The  $j$ -dimensional sources and targets of this  $\mathcal{I}^n$  are themselves blocks, acting on the piles corresponding to the still lower dimensional sources and targets.

Our immediate concern is to develop an appropriate formalism and notation to make these ideas concise for the  $n$ -cube. We take an inductive approach starting with the raw material above.

For  $\mathcal{I}^3$  we saw that there are three  $\mathcal{I}^2$ s making up the 2-dimensional source. If we ignore the order in which these cubes are written, as well as their position in  $\mathcal{I}^3$ , we have a set of  $\mathcal{I}^2$ s we shall call the *source 2-face* of  $\mathcal{I}^3$ . More generally we will define the source and target  $k$ -faces of  $\mathcal{I}^n$ , denoted  $\psi_k(n)$  and  $\omega_k(n)$  respectively. For given  $n$ , we obtain two sequences of sets  $\psi_0(n), \dots, \psi_{n-1}(n)$ ,  $\psi_n(n)$ , and  $\omega_0(n), \dots, \omega_{n-1}(n)$ ,  $\omega_n(n)$ . The subscript indicates the dimension of elements of the set.

**Definition 4.1:** Let  $\psi_0[n] = \sigma_0[n]$ ,  $\omega_0[n] = \tau_0[n]$  and  $\psi_j[n] = [n] = \omega_j[n]$ , for each  $j \geq n$ . Inductively define  
a) For  $k$  even:

$$\begin{aligned}\psi_k[n] &= \mu\psi_{k-1}[n-1] \cup \lambda\psi_k[n-1] \\ \omega_k[n] &= \nu\omega_{k-1}[n-1] \cup \mu\omega_{k-1}[n-1]\end{aligned}$$

b) For  $k$  odd:

$$\begin{aligned}\psi_k[n] &= \mu\psi_{k-1}[n-1] \cup \nu\psi_k[n-1] \\ \omega_k[n] &= \lambda\omega_{k-1}[n-1] \cup \mu\omega_{k-1}[n-1]\end{aligned}$$

We have deliberately written these as (inductively) ordered, since the order reflects their sequence of operation on piles. A simple induction from the motivational examples proves

**Theorem 4.2:** (i) Identifying elements of either  $\psi_k[n]$  or  $\omega_k[n]$  along common boundaries produces a  $k$ -dimensional disk  $D_k$ .  
(ii)  $\partial D_k[n] = \psi_{k-1}[n] \cup \omega_{k-1}[n] = \partial\omega_k[n]$ .

**Examples 4.3:** In the following table we list the source and target faces for  $n = 1, \dots, 4$ .

[n]	$\psi_0$	$\omega_0$	$\psi_1$	$\omega_1$	$\psi_2$	$\omega_2$	$\psi_3$	$\omega_3$	$\psi_4$	$\omega_4$
[1]	—	+	0	0	0	0	0	0	0	0
[2]	--	++	-0 0 +	0 — +0	0 0	0 0	0 0	0 0	0 0	0 0
[3]	---	+++	--0 -0 + 0 ++	0 --- +0 — ++0	-0 0 0 +0 0 0 —	0 0 + 0 —0 +0 0	0 0 0	0 0 0	0 0 0	0 0 0
[4]	----	++++	---0 --0 + -0 ++ 0 +++	0 ---- +0 --- ++0 — +++0	---0 0 -0 +0 0 ++0 -0 0 — 0 +0 — 0 0 ---	0 0 ++ 0 —0 + +0 0 + 0 —0 +0 —0 ++0 0	-0 0 0 0 +0 0 0 0 —0 0 —0 0 0 0 0 +	0 0 0 — 0 0 +0 0 —0 0 +0 0 0	0 0 0 0	0 0 0 0

Denote by  $C_k^n$  the binomial coefficients arising in Pascal's triangle.

**Definition 4.4:** An  $(m, n)$ -pile is a sequence of  $(m + 1)$  sets  $v_j$ ,  $j = 0, \dots, m$  such that

- (a)  $v_j \subset \mathcal{I}^n$
- (b) each  $v_j$  has cardinality  $C_j^m$
- (c) each  $x \in v_j$  has dimension  $j$ .

Fix  $n$ , and denote the collections of source and target faces by

$$\Psi_n = \cup \psi_k[n]$$

$$\Omega_n = \cup \omega_k[n]$$

**Proposition 4.5:**  $\Psi_n$  and  $\Omega_n$  are  $(n, n)$ -piles.

**Remark 4.6:** Each sub- $k$ -cube of the  $n$ -cube is parallel to a subspace of  $R^n$ . The subspaces arising in this fashion correspond to choices of  $k$  elements of  $\{1, \dots, n\}$  as subscripts for coordinate axes. Thus there are  $C_k^n$  possible such subspaces, and for each,  $2^{n-k}$  parallel copies in the boundary of  $\mathcal{I}^n$ . We shall call the collection of all such parallel copies a *parallel set*. Observe that the inductive definition of source and target faces canonically chooses a representative from each parallel set.

**Definition 4.7:** We will call any choice of a unique representative from each parallel set of  $k$ -dimensional subcubes a *k-section*.

Hence Theorem 4.3 tells us that there are at least two distinct  $k$ -sections whose topological union is a disc. Moreover, these unions are coherent across dimensions.

Note also that differential  $p$ -forms in  $n$ -space admit a basis analogous to a  $p$ -section. This suggests some relationship to de Rham cohomology.

For each  $x \in \mathcal{I}^n$ ,  $|x| = p$ , we define  $\Psi_x$  and  $\Omega_x$  as  $(p, n)$ -piles by using  $x$  to embed elements of  $\Psi_p$  and  $\Omega_p$  into  $\mathcal{I}^n$ .

The definitions given above lead to the structure referred to in the introduction. We describe the relationship between the source- and target- $n$ -piles, and how these relate to the cocycle conditions.

**Proposition 4.8:** For each  $k$ ,  $\psi_k[n] = -\omega_k[n]$ , where the antipodal map  $- : \mathcal{I}^n \rightarrow \mathcal{I}^n$  is the involution sending a word  $x$  to  $-x$ , obtained from  $x$  by interchanging each  $-$  and  $+$ . Hence  $\psi_k[n]$  and  $\omega_k[n]$  are interchanged by the antipodal map on  $[n]$ .

**Remark 4.9:** Since the parallel set to which a sub-cube belongs is determined by the position of 0's in its word form, the antipodal map preserves parallel sets.

If now we have an  $(n, n)$ -pile  $\Pi$  into which there exists an embedding  $\Psi_x$  for some  $x \in \mathcal{I}^n$ , we obtain a new list by replacing  $\Psi_x$  by  $\Omega_x$ . Note that if  $|x| = j$ ,  $x \in \psi_j[n]$ . We have a map  $\pi_x$ , whose source and target are  $(n, n)$ -piles. The top dimensional cube of  $x$  (merely  $x$  itself) remains in the pile, only its lower dimensional faces changing. (Here we abuse notation somewhat as  $\Psi_x$  may occur in many different  $(n, n)$ -piles.)

Denote by  $*_k$  the operation which applies the involution “ $-$ ” to every element of  $v_j$ ,  $j \leq k - 1$ . The operation  $*_0$  will denote the identity. If now we begin with a pile  $\Pi_0$  and an embedding  $\Psi_{x_0} \hookrightarrow \Pi_0$ , it may be possible to find sequences

$$x_0, *_0, x_1, *_1, \dots, x_j, *_j$$

such that a sequence of piles is generated from  $\Pi_0$ , each obtained from the previous by applying consecutively the  $\pi_{x_i}$  and  $*_{i_k}$ . Observing that  $\pi_x$  is parallel set preserving, it is not difficult to prove

**Proposition 4.10:** If  $\Pi_0$  is a pile for which each  $v_k$  is a  $k$ -section whose topological union, identifying along common boundaries, is a disk with boundary invariant under the antipodal map, the same is true for  $*_j(\Pi_0)$  or  $\pi_x(\Pi_0)$  whenever these are defined. Furthermore, the boundary of each such disc remains the same.

The  $(n - 1)$ -source of  $[n]$ , denoted  $\sigma_{n-1}[n]$ , will be such a sequence containing  $\psi_{n-1}[n]$ , such that the composite of the sequence has source  $\Psi_n$  and target  $\mathcal{T}_n = \Omega_n$  but with  $\psi_{n-1}$  replacing  $\omega_{n-1}$ . By applying the involution “ $-$ ” to the sequence written in reverse order we will find another such sequence, which we call the  $(n - 1)$ -target  $\tau_{n-1}[n]$  of  $[n]$ . This has source pile  $\Psi_n$  with  $\psi_{n-1}$  replaced by  $\omega_{n-1}$ . The sequence uses each element of  $\omega_{n-1}[n]$ , and has target  $\Omega_n$ .

**Definition 4.11:** The formal equality  $\sigma_{n-1}[n] = \tau_{n-1}[n]$  is called the  $(n - 1)$ -cocycle condition.

## 5 Modifying piles - Blocks

Intuitively a pile describes a nested set of disks  $V_k$  of descending dimension, geometrically embedded in the  $n$ -cube.  $V_{n-2}$  is the  $(n - 2)$ -source in some stage of (cubical) isotopy or deformation across the interior of  $V_{n-1}$ , keeping the boundary fixed. The disc  $V_{n-1}$  in practice is either the  $(n - 1)$ -source or target of  $\mathcal{I}^n$ . In turn each  $V_{k-1}$  is the  $(k - 1)$ -source of  $\mathcal{I}^n$  after some deformation across  $V_k$ , keeping the boundary fixed. Each disk  $V_k$  decomposes as a set  $v_k$ , the list of “currently encountered”  $k$ -dimensional faces of  $\mathcal{I}^n$  as we deform across the cubes of  $v_{k+1}$ . (See Figure 8.)

Suppose at some stage we have a pile  $\Pi$  such that  $v_j = \psi_j[n]$  for  $j \leq (k - 2)$ , and wish to apply  $\pi_x$  for  $x \in v_k$ . This requires that the  $(k - 1)$ -,  $(k - 2)$ -, ..., 0-faces of  $x$  appear in the appropriate  $v_j$ . In general this will not be the case. By looking at  $v_{k-1}$  we can decide if there is any chance of applying  $\pi_x$ . If  $x$  has  $(k - 1)$  source faces appearing in  $v_{k-1}$ , it should be possible to use the remaining  $(k - 1)$ -cubes in  $v_{k-1}$  to change the lower  $v_j$ 's until the rest of the source faces of  $x$  appear. We illustrate this schematically in Figure 9. The  $(k - 1)$ -source cubes of  $x$  are those shaded among the cubes of  $v_{k-1}$ .

The heavy path depicts  $v_{k-2}$ , from which it is clear the  $(k - 2)$ -source faces of  $x$  do not all appear. Hence we apply consecutively  $\pi_\alpha$ ,  $\pi_\beta$ , then  $\pi_x$ , and finally  $\pi_\gamma$  to obtain a pile with  $v_{k-2} = t\psi_{k-2}$  replaced by  $\omega_{k-2}$ . If we now apply  $*_{k-1}$ , having used all of the faces of  $v_{k-1}$  in the sequence of applications, we can replace  $\omega_{k-2}$  by  $\psi_{k-2}$  and try to find the next cube  $x'$  higher up the list to apply. We illustrate in Figure 10.

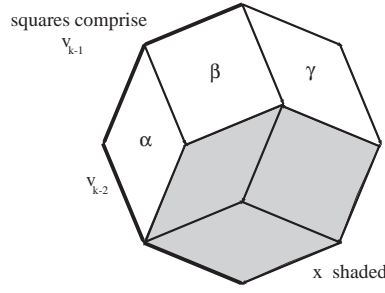


Figure 9: The 2-cubes  $\alpha$ ,  $\beta$  can be used to move the 1-source of [4] to include the 1-source of  $x$ .

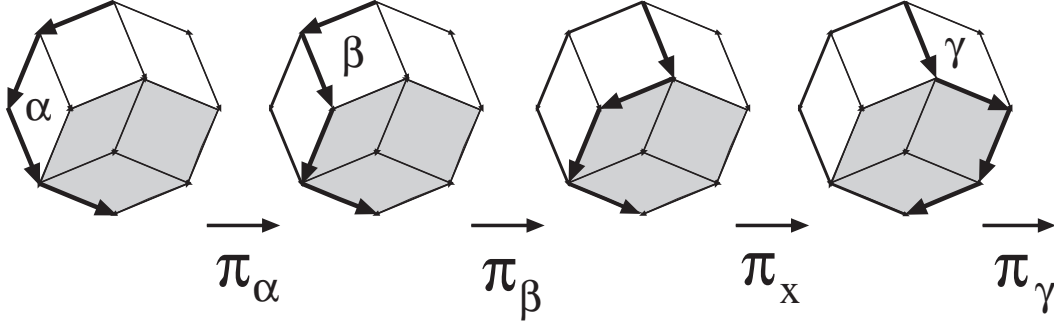


Figure 10: Lower-dimensional completion of a cell deformation: using 2-cubes to move the 1-source to include the 1-source of a higher 3-cube  $x$ , so that  $\pi_x$  can be applied.

**Remark 5.1:** There are two things to note here. The first is that to apply  $\pi_\alpha$  all of  $\alpha$ 's lower dimensional source faces must appear in the pile. We may first have to apply the same process as above, at a lower dimension. Secondly we have the “moral” of the heuristic description: to apply  $\pi_x$ , for  $x \in v_k$ , we must work in conjunction with the remaining  $\pi_\alpha$ s of  $v_{k-1}$ . Hence  $x$  occurs as part of a “block” of cubes, namely  $x$  and the rest of the cubes in  $v_{k-1}$  which are not in  $x$ . This “block” is on the one hand derived from the pile, and on the other, composed of cubes which serve to modify the pile so that the process can continue. The conclusion is that every time we apply  $\pi_x$ ,  $x \in v_k$ , there is an associated “sweep” of the  $(k-2)$ -source of  $\mathcal{I}^n$  across to the  $(k-2)$ -target, at some stage sweeping across  $x$ . All of these “sweeps” occur at different dimensions simultaneously.

We can try to find an order in which to proceed directly from the pile  $\psi_n$ , in order to use all of the elements of  $\psi_{n-1}$  to change each lower  $\psi_k$  into the corresponding  $\omega_k$ . This order will not be unique. However, knowing an order from the previous dimension will enable us to make inductively a natural choice.

Note that the effect of  $\pi_x$ ,  $|x| = k$ , on  $v_j$ ,  $j \leq k$  can be decomposed :  $\pi_x$  alters  $C_k^j$  elements, an effect which can be obtained in  $C_k^{k-1}$  stages corresponding to the effect of each  $(k-1)$ -source-face of  $x$  applied consecutively (and similarly target faces). Each of these can be further decomposed, all the way down to dimension  $j+1$ . In Figure 11 we break up the effect of  $\pi_x$  of the previous figure.

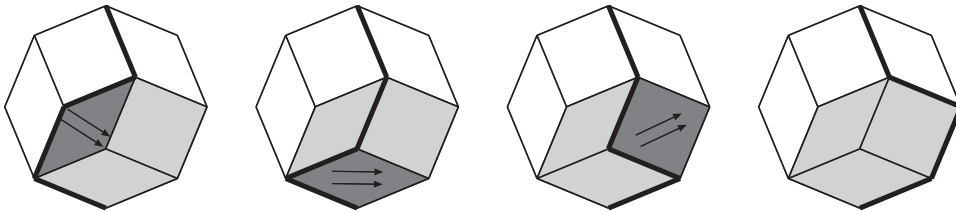


Figure 11: Decomposition of the action of a  $k$ -cell into the action of  $(k-1)$ -cells



**Definition 5.6:** By a  $j$ -sub-block of a  $k$ -block we will mean any sub-block characterized by elements between a pair  $*_r, *_s$ , for  $r, s \geq j - 1$ , such that no  $*_l$  occurs between them for  $l \geq j$ . A full  $j$ -sub-block will be a  $j$ -sub-block maximal with respect to this condition.

For each word  $x$  above, replace  $x$  in the string by  $\pi_x$ . Interpret each  $*_i$  as “rewrite  $\omega_j[n]$  as  $\psi_j[n]$  for each  $j \leq (i - 1)$ ”, where  $n$  is the word length.

For (c) we obtain the sequence of piles in the following Table 2. The 12 columns correspond to the geometric figures in Figure 12. Those on the left and right are those of  $\Psi_3$  and  $\Omega_3$ . The elements of each  $v_k$  are shown in bold, and the hatched element is the appropriate  $x$  for which  $\pi_x$  is about to be applied. This illustrates the “nestings” of disks referred to earlier.

**Proposition 5.7:** The 0, 1-, 2-, 3-cocycle conditions are given by (1)–(4) above.

$-0\ 0$	$*_0$	$0\ ++$	$*_1$	$-0\ -$	$*_0$	$0\ ++$	$*_1$	$0\ 0\ -$	$*_0$	$++0$	
---	-++	-++	+++	---	-+-	-+-	+++	---	++-	++-	+++
- - 0	- + 0	- + 0	- + 0	- + 0	- + 0	- + 0	++ 0	++ 0	++ 0	++ 0	++ 0
- 0 +	- 0 -	- 0 -	- 0 -	- 0 -	- 0 -	- 0 -	- 0 -	- 0 -	+ 0 -	+ 0 -	+ 0 -
0 ++	0 ++	0 ++	0 ++	0 ++	0 ++	0 ++	0 +-	0 +-	0 --	0 --	0 --
- 0 0	- 0 0	- 0 0	- 0 0	- 0 0	- 0 0	- 0 0	- 0 0	- 0 0	- 0 0	- 0 0	- 0 0
0 + 0	0 + 0	0 + 0	0 + 0	0 + 0	0 + 0	0 + 0	0 + 0	0 + 0	0 + 0	0 + 0	0 + 0
0 0 -	0 0 -	0 0 -	0 0 -	0 0 -	0 0 -	0 0 -	0 0 -	0 0 -	0 0 -	0 0 -	0 0 -
0 0 0	0 0 0	0 0 0	0 0 0	0 0 0	0 0 0	0 0 0	0 0 0	0 0 0	0 0 0	0 0 0	0 0 0

**Table 2 - the 2-cocycle source of the 3-cube**

We can tabulate part of the effect of the block (d) on  $\psi_4$  by recording the changes at the 2-dimensional level. (We can always ignore the changes lower down in the pile to save space in description.) Since the highest dimension of  $x$  occurring for any application of  $\pi_x$  is 3, the set  $\psi_3$  remains unaltered. The reader may wish to refer forward to the “octagon of octagons” depicted in Figure 7, comparing the lists of 2-cubes above with the left hand side configurations in the figure.

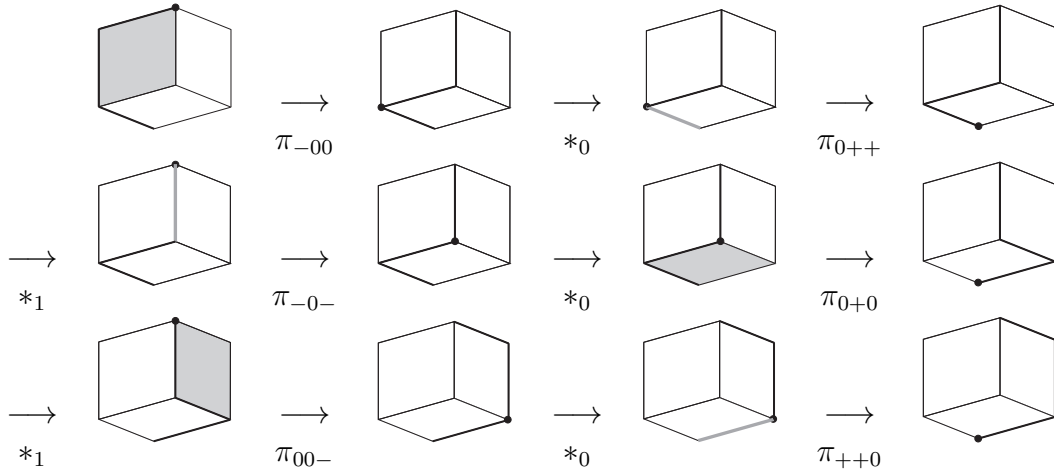


Figure 12: Rewriting: the action of a 2-block on unions of lower-dimensional subdiscs

$-0\ 0\ 0$	$0\ +0\ 0$	$0\ 0\ -0$	$0\ 0\ 0\ +$
$--0\ 0$	$-+0\ 0$	$++0\ 0$	$++0\ 0$
$-0\ +0$	$-0\ -0$	$-0\ -0$	$+0\ -0$
$0\ ++0$	$0\ ++0$	$0\ + -0$	$0\ --0$
$-0\ 0\ -$	$-0\ 0\ +$	$-0\ 0\ +$	$-0\ 0\ +$
$0\ +0\ -$	$0\ +0\ -$	$0\ +0\ +$	$0\ +0\ +$
$0\ 0\ --$	$0\ 0\ --$	$0\ 0\ --$	$0\ 0\ -+$
			$0\ 0\ ++$

**Table 3 – the source 3-cube action on source 2-cubes of the 4-cube**

We denote by  $B_n^k$  the set of  $k$ -blocks of elements of  $\mathcal{I}^n$ . Note that there are natural inclusions  $B_n^k \rightarrow B_n^{k+1}$ , and an isomorphism  $\mathcal{I}^n \cong B_n^0$ . The  $B_n^k$  can be used to define a structure analogous to the  $N_\omega$  occurring in Street [9]. Note that there is a “forgetful” map which sends a  $k$ -block  $\alpha$  to its underlying set of elements of  $\mathcal{I}^n$ . We can thus define the *dimension* of  $\alpha$  to be the maximum of the dimensions of its elements.

**Definition 5.8:** A *distinguished*  $k$ -block is a  $k$ -block for which each sub- $j$ -block has all but one sub- $(j-1)$ -blocks of dimension  $j$ , the other  $(j-1)$ -block of dimension  $\geq j$ . We will denote by  $D_n^k \subset B_n^k$  the subset of distinguished blocks. A 0-block is always considered distinguished.

**Examples 5.9:** All of the examples given above are distinguished. Our inductive definition of cocycles will demonstrate that all cocycles are distinguished. The notion makes precise how an element  $x$  of a pile can only be applied in conjunction with those elements of a pile of the next lower dimension which do not occur as source faces of  $x$ . Thus  $x$  must be “stretched out” appropriately to the source and target of the whole pile.

Writing out the terms using the  $*_i$  provides not only the order in which to modify the  $(n, n)$ -piles, but also tells us at which stage we must rewrite the pile from below the  $i$ -dimensional level. Moreover, a  $*_k$  separating two blocks  $\alpha_i$  and  $\alpha_{i+1}$  indicates that geometrically the respective collections of cubes are “glued together” along a  $k$ -dimensional disk  $D$ .  $D$  itself is a collection of subcubes, and has a pile/block description as the  $k$ -target of  $\alpha_i$  and the  $k$ -source of  $\alpha_{i+1}$ .

## 6 Sources and targets - inductive cocycles

We use the order defining the  $(n-2)$ -source of  $\mathcal{I}^{n-1}$  to obtain the  $(n-1)$ -source of  $\mathcal{I}^n$ . Since  $\mathcal{I}^n$  is obtained geometrically by thickening the  $(n-1)$ -cube, all of the cubes in the expression of the  $(n-2)$ -source of  $\mathcal{I}^{n-1}$  will play a role in the form of either old, new or thickened copies, via the sources and targets they themselves have in lower dimensions. All  $k$ -blocks subsequently considered will be distinguished, unless otherwise stated. Our intention is to describe how to complete the table for all  $n, \sigma_k$ :

	$\sigma_0[n]$	$\sigma_1[n]$	$\sigma_2[n]$	$\sigma_3[n]$
[1]	—	0	0	0
[2]	--	$\begin{smallmatrix} -0 \\ 0+ \end{smallmatrix}$	0 0	0 0
[3]	---	$\begin{smallmatrix} --0 \\ -0+ \\ 0++ \end{smallmatrix}$	$\begin{smallmatrix} -0\ 0 & -0\ - & 0\ 0\ - \\ 0\ ++ & 0\ +0 & ++0 \end{smallmatrix}$	0 0 0
[4]	----	$\begin{smallmatrix} ---0 \\ --0+ \\ -0++ \\ 0+++ \end{smallmatrix}$	$\begin{smallmatrix} --0\ 0 & --0\ - & --0\ - & -0\ 0\ - & -0\ - - & 0\ 0\ - - \\ -0\ ++ & -0\ +0 & -0\ + - & 0\ + + - & 0\ +0\ - & ++0\ - \\ 0\ +++ & 0\ +++ & 0\ +++ & +++0 & +++0 & +++0 \end{smallmatrix}$	$\begin{smallmatrix} -0\ 0\ 0 & -0\ - - & -0\ - - & 0\ 0\ - - \\ 0\ +++ & -+0\ - & 0\ +0\ - & ++0\ - \\ & 0\ +++0 & +++0 & +++0 \end{smallmatrix}$ $\begin{smallmatrix} ---0 & -0\ -0 & -0\ - - & 0\ 0\ - - \\ -0\ 0\ + & -+0\ + & 0\ +0\ 0 & ++0\ - \\ 0\ +++ & 0\ +++ & & +++0 \end{smallmatrix}$ $\begin{smallmatrix} ---0 & ---0 & 0\ 0\ -0 & 0\ - - - \\ -0\ 0\ + & -0\ - + & ++0\ + & +0\ - - \\ 0\ +++ & 0\ +0\ + & & +++0\ 0 \end{smallmatrix}$ $\begin{smallmatrix} ---0 & 0\ - -0 & 0\ - - - & 0\ - - - \\ 0\ 0\ 0\ + & +0\ - + & +0\ -0 & +0\ - - \\ & ++0\ + & ++0\ + & +++0\ 0 \end{smallmatrix}$

We shall only need to consider  $\sigma_k[n]$ ,  $k < n$ . It is easy to list the sub- $k$ -cubes of  $\mathcal{I}^n$  which occur, but the tricky part is to pin down and order those lower dimensional subcubes which stretch out the  $\mathcal{I}^k$ s to  $\sigma_j[n]$  and  $\tau_j[n]$  for  $j < k$ .

We shall inductively define functions

$$\begin{aligned} \sigma_k, \tau_k &: D_n^m \longrightarrow D_n^k \\ \lambda_k, \nu_k &: D_n^k \longrightarrow D_{n+1}^k \\ \mu_k &: D_n^k \longrightarrow D_{n+1}^{k+1} \end{aligned}$$

- (a) For the  $n$ -cube  $[n]$ , define  $\sigma_n[n] = [n] = \tau_n[n]$ . Hence these are 0-blocks (and  $k$ -blocks by default.)
- (b) For 0-blocks, define maps  $\mu_0, \nu_0$  and  $\lambda_0 : D_n^0 \rightarrow D_{n+1}^0$  by  $x \mapsto \mu x, \nu x, \lambda x$  respectively. Extend the definition to an arbitrary  $k$ -block by operating on each sub-0-block. (In practice symbolically this will partly involve adding  $-, 0, +$ .)
- (c) Define  $\sigma_0[n] = \lambda_0(\sigma_0[n-1])$ , and  $\tau_0[n] = \nu_0(\tau_0[n-1])$ . This is the geometrically obvious choice. Extend the definition to  $\mathcal{I}^n = D_n^0$ , as described using  $*$ . For an arbitrary  $k$ -block, define  $\sigma_0$  and  $\tau_0$  by restricting to the first 0-block.
- (d) Define the 1-blocks  $\sigma_1[n]$  and  $\tau_1[n]$  inductively by

$$\begin{aligned} \sigma_1[n] &= \begin{smallmatrix} \mu_0\sigma_0[n-1] \\ \nu_0\sigma_1[n-1] \end{smallmatrix} & \tau_1[n] &= \begin{smallmatrix} \lambda_0\tau_1[n-1] \\ \mu_0\tau_0[n-1] \end{smallmatrix} \end{aligned}$$

These coincide with  $\psi_1[n]$  and  $\omega_1[n]$  respectively. Extend the definition of  $\sigma_1$  and  $\tau_1$  to  $D_n^0 \rightarrow D_{n+1}^1$ . For  $\alpha \in D_n^1$ , set

$$\sigma_1(\alpha) = \sigma_1 \left| \begin{smallmatrix} \alpha_1 \\ \vdots \\ \alpha_t \end{smallmatrix} \right| := \left| \begin{smallmatrix} \sigma_1\alpha_1 \\ \vdots \\ \sigma_1\alpha_t \end{smallmatrix} \right| \qquad \tau_1(\alpha) = \tau_1 \left| \begin{smallmatrix} \alpha_1 \\ \vdots \\ \alpha_t \end{smallmatrix} \right| := \left| \begin{smallmatrix} \tau_1\alpha_1 \\ \vdots \\ \tau_1\alpha_t \end{smallmatrix} \right|$$

This means that all of the columns are to be strung together, one on top of the next. For  $\alpha \in D_n^k$ , we define the 1-source and 1-target by restricting to the first column. Geometrically think of the column as a string of subcubes joined together along their 0-sources and 0-targets. Taking the 1-source then strings together the  $\mathcal{I}^1$ s making up the 1-sources of the elements of the initial column. Note that for an arbitrary 1-block, 0-sources and 0-targets will not match, but those with which we are inductively concerned do continue to enjoy this property.

(e) Define  $\lambda_1$  and  $\nu_1$ :  $D_n^1 \rightarrow D_{n+1}^1$  by

$$\lambda_1(\alpha) = \lambda_1 \begin{vmatrix} \alpha_1 \\ \vdots \\ \alpha_t \end{vmatrix} := \begin{vmatrix} \lambda_0 \alpha_1 \\ \vdots \\ \lambda_0 \alpha_t \\ \mu_0 \tau_0[n] \end{vmatrix} \quad \nu_1(\alpha) = \nu_1 \begin{vmatrix} \alpha_1 \\ \vdots \\ \alpha_t \end{vmatrix} := \begin{vmatrix} \mu_0 \sigma_0[n] \\ \nu_0 \alpha_1 \\ \vdots \\ \nu_0 \alpha_t \end{vmatrix}$$

Hence  $\sigma_1[n] = \nu_1 \sigma_1[n-1]$  and  $\tau_1[n] = \lambda_1 \tau_1[n-1]$ . Extend the domain of  $\lambda_1$  and  $\nu_1$  to  $D_n^k$  by application to each sub-1-block. Note that  $\mu_0 \tau_0$  and  $\mu_0 \sigma_0$  “stretch out” the ends of a string of  $\mathcal{I}^1$ s arising from  $\mathcal{I}^{n-1}$ , which stretched between the 0-source and 0-target of  $\mathcal{I}^{n-1}$ , to the 0-source and 0-target of  $\mathcal{I}^n$ .

(f) Now define  $\mu_1 : D_n^1 \rightarrow D_{n+1}^2$  by

$$\mu_1(\alpha) = \mu_1 \begin{vmatrix} \alpha_1 \\ \vdots \\ \alpha_t \end{vmatrix} := \begin{vmatrix} \mu_0 \alpha_1 \\ \nu_0 \sigma_1 \alpha_2 \\ \vdots \\ \nu_0 \sigma_1 \alpha_t \end{vmatrix} \begin{vmatrix} \lambda_0 \tau_1 \alpha_1 \\ \mu_0 \alpha_2 \\ \vdots \\ \nu_0 \sigma_1 \alpha_t \end{vmatrix} \cdots \begin{vmatrix} \lambda_0 \tau_1 \alpha_1 \\ \lambda_0 \tau_1 \alpha_2 \\ \vdots \\ \mu_0 \alpha_t \end{vmatrix}$$

This is illustrated in Figure 13.

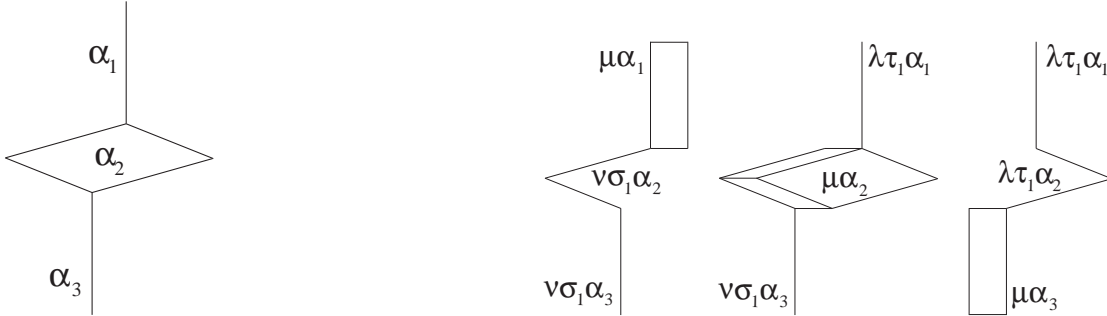


Figure 13: The source thickens

(g) To define  $\sigma_j$  and  $\tau_j$  on  $D_n^k$  we use triple induction, first on  $n$ , then  $k$ , then  $j$ . For  $j = n$ ,  $k = 0$ , we have already given the definition for “pure” cubes. Extension to  $D_n^0$  is carried out as above.

All that is required is to avoid loops in our definitions. For  $\sigma_2$  we extend the domain to  $D_n^0$  by consecutively defining  $\sigma_2[2]$ ,  $\tau_2[2]$ , and

$$\sigma_2[n] = \mu_1 \sigma_1[n-1] | \lambda_1(\sigma_2[n-1]); \quad \tau_2[n] = \nu_1(\tau_1[n-1]) | \mu_1 \tau_1[n-1].$$

This makes precise the way in which the 1-dimensional sources and targets of  $\mathcal{I}^{n-1}$  contribute to the 2-dimensional sources of  $\mathcal{I}^n$ .

We now wish to extend the domain to  $D_n^1$ . It is here that the notion of distinguished k-block is crucial. In each 1-block, every sub-0-block except possibly one,  $\alpha_i$ , say, has dimension 1. These sub-blocks have trivial 2-dimensional sources, and serve to connect the 0-sources of the 1-block to the 0-source of  $\alpha_i$ , and similarly for the 0-targets of  $\alpha$  and  $\alpha_i$ . By induction,  $\alpha_i$  has sources and targets in lower dimensions. For  $\sigma_2(\alpha_i)$  we obtain a 2-block, each 1-block of which stretches between the 0-source and 0-target of  $\alpha_i$ . In order to define  $\sigma_2$  of the 1-block  $\alpha$ , we must stretch out each of the 1-blocks of  $\sigma_2(\alpha_i)$  by adding to each column a copy of the preceding  $\alpha_j$ s of  $\alpha$  at the top, and those succeeding at the bottom. We illustrate this in Figure 14.

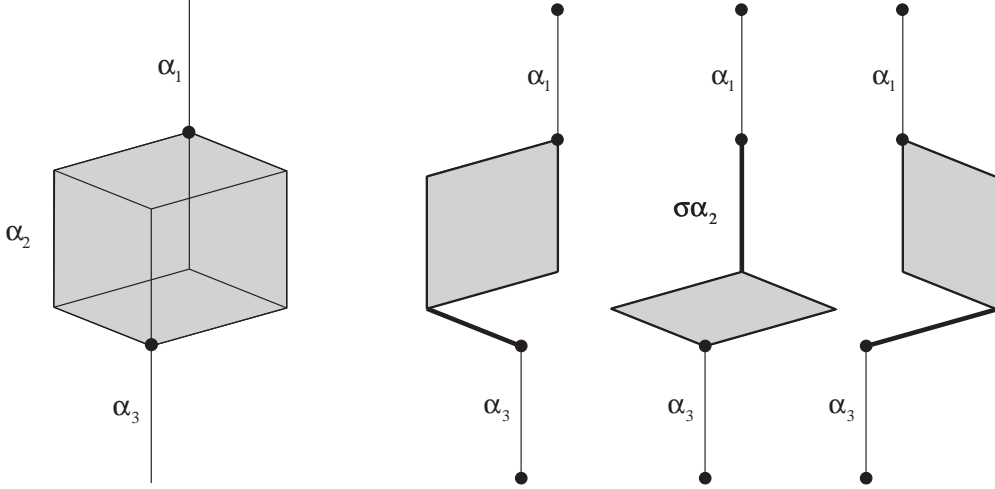


Figure 14: Stretching out lower dimensional sources and targets of distinguished blocks :  $\alpha_2 \rightarrow \sigma_2 \alpha_2$ . Left is the block with distinguished 3-cube; on the right, three 2-cubes are stretched out.

Note that the resulting 2-block is again distinguished. The same procedure will be repeated in higher dimensions. Thus we extend the domain of  $\sigma_2$  to  $D_n^1$  by setting

$$\sigma_2(\alpha) = \sigma_2 \begin{vmatrix} \alpha_1 \\ \vdots \\ \alpha_i \\ \vdots \\ \alpha_t \end{vmatrix} := \begin{vmatrix} \alpha_1^{(m)} \\ \vdots \\ \sigma_2 \alpha_i \\ \vdots \\ \alpha_t^{(m)} \end{vmatrix}$$

Here we have written the superscript “ $m$ ” to indicate multiple copies will be taken, since  $\sigma_2(\alpha_i)$  is a 2-block. This is 2-category pasting again.

To define  $\sigma_2$  on 2-blocks, apply it as defined above consecutively to each sub-1-block, juxtaposing the 2-blocks so obtained from left to right in the original order of the 1-blocks. For  $\alpha \in D_n^k$ ,  $k \geq 2$ , define  $\sigma_2(\alpha)$  to be  $\sigma_2$  of the first sub-2-block.

- (h) We now presume all maps defined for subscripts  $j < k < n$ , on all  $D_n^i$ . To define  $\sigma_k, \tau_k : D_n^k \rightarrow D_{n+1}^k$ : For  $k$  even, define

$$\sigma_k[n] = \mu_{k-1} \sigma_{k-1}[n-1] \mid \lambda_{k-1} \sigma_k[n-1]; \quad \tau_k[n] = \nu_{k-1} \tau_k[n-1] \mid \mu_{k-1} \tau_{k-1}[n-1]$$

For  $k$  odd we set

$$\sigma_k[n] = \begin{vmatrix} \mu_{k-1} \sigma_{k-1}[n-1] \\ \lambda_{k-1} \sigma_k[n-1] \end{vmatrix}; \quad \tau_k[n] = \begin{vmatrix} \nu_{k-1} \tau_k[n-1] \\ \mu_{k-1} \tau_{k-1}[n-1] \end{vmatrix}$$

This allows us to define  $\sigma_k[n]$  for all blocks after we set  $\sigma_n[n] = \tau_n[n] := [n]$  for  $k \geq n$ . To extend the definition to higher blocks, we use the intuition we have of the underlying geometry. A  $k$ -block corresponds to a possibly higher dimensional cube  $x$ , together with a number of others of dimension  $k$ , each “filled out” by lower dimensional cubes. Then  $\sigma_k$  and  $\tau_k$  should correspond to the  $k$ -cubes already present, together with  $\sigma_k$  or  $\tau_k$  of  $x$ , with everything still appropriately stretched out by lower dimensional cubes. We merely break up the action of  $x$  on the pile sets  $v_j$ ,  $j < k$ , as the composition of the action of its  $k$ -dimensional faces.

Specifically, to define  $\sigma_k(\alpha)$ , where  $\alpha$  is a  $j$ -block,

( $h_1$ ) If  $j \geq k$ , define  $\sigma_k(\alpha)$  to be  $\sigma_k$  applied to the first  $k$ -block of  $\alpha$ .

( $h_2$ ) For  $j = k$ , define  $\sigma_k$  to be the juxtaposition of  $\sigma_k$  applied to the consecutive  $(k-1)$ -blocks of  $\alpha$ . Hence we need only be specific for  $j \leq k-1$ .

( $h_3$ ) If  $\alpha$  has dimension  $k$ , we set  $\sigma_k(\alpha) := \alpha$ . Otherwise there is a 0-block contained in a unique sub- $(k-1)$ -block  $\gamma$  of dimension  $m > k$ . Since  $\sigma_k(\gamma)$  is a  $k$ -block, we stretch it out by the same sub-blocks which stretch out  $\gamma$  inside  $\alpha$ . Now  $\gamma$  is contained in a 1-block, with 0-blocks above and below. We add these 0-blocks above and below every 1-block of  $\sigma_k(\alpha)$ .

( $h_4$ ) Similarly,  $\gamma$  occurs in a unique 2-block, with 1-blocks occurring before and after the 1-block in which  $\gamma$  occurs. We thus add each of these 1-blocks appropriately before and after every 1-block of  $\sigma_k(\gamma)$ .

( $h_5$ ) Proceeding in this way, up to the highest dimensional block, we fill out  $\sigma_k(\gamma)$  to obtain the  $k$ -block we define to be  $\sigma_k(\alpha)$ .

The procedure/definition merely tells us that if an element  $x$  has been stretched out by lower dimensional cubes, we use these to stretch out the components of  $x$ .

$$\sigma_k(\alpha) = \sigma_k \left| \begin{array}{c} \alpha_1 \\ \vdots \\ \alpha_i \\ \vdots \\ \alpha_t \end{array} \right| := \left| \begin{array}{c} \alpha_1^{(m)} \\ \vdots \\ \sigma_k \alpha_i \\ \vdots \\ \alpha_t^{(m)} \end{array} \right|$$

where again  $\alpha_i$  is the distinguished sub-block, and repetitions are required for the other blocks. We proceed similarly for  $\tau_k$  and for the case  $k$  even.

(i) Now define, for  $k$  odd,  $\lambda_k$  and  $\nu_k : D_n^k \rightarrow D_{n+1}^k$  by

$$\lambda_k(\alpha) = \lambda_k \left| \begin{array}{c} \alpha_1 \\ \vdots \\ \alpha_t \end{array} \right| := \left| \begin{array}{c} \lambda_{k-1} \alpha_1 \\ \vdots \\ \lambda_{k-1} \alpha_t \\ \mu_{k-1} \tau_{k-1}[n] \end{array} \right| ; \quad \nu_k(\alpha) = \nu_k \left| \begin{array}{c} \alpha_1 \\ \vdots \\ \alpha_t \end{array} \right| := \left| \begin{array}{c} \mu_{k-1} \sigma_{k-1}[n] \\ \nu_{k-1} \alpha_1 \\ \vdots \\ \nu_{k-1} \alpha_t \end{array} \right|$$

For  $k$  even, set

$$\begin{aligned} \nu_k(\alpha) &= \nu_k \left| \alpha_1 \cdots \alpha_t \right| := \left| \nu_{k-1} \alpha_1 \mid \cdots \mid \nu_{k-1} \alpha_t \mid \mu_{k-1} \tau_{k-1}[n] \right| \\ \lambda_k(\alpha) &= \lambda_k \left| \alpha_1 \cdots \alpha_t \right| := \left| \mu_{k-1} \sigma_{k-1}[n] \mid \lambda_{k-1} \alpha_1 \mid \cdots \mid \lambda_{k-1} \alpha_t \right| \end{aligned}$$

(j) Now define for  $k$  odd

$$\mu_k(\alpha) = \mu_k \left| \begin{array}{c} \alpha_1 \\ \vdots \\ \alpha_t \end{array} \right| := \left| \begin{array}{c} \frac{\mu_{k-1}\alpha_1}{\nu_{k-1}\sigma_k\alpha_2} \\ \vdots \\ \frac{\mu_{k-1}\sigma_k\alpha_t}{\nu_{k-1}\sigma_k\alpha_t} \end{array} \right| \left| \begin{array}{c} \frac{\lambda_{k-1}\tau_k\alpha_1}{\mu_{k-1}\alpha_2} \\ \vdots \\ \frac{\lambda_{k-1}\sigma_k\alpha_t}{\mu_{k-1}\alpha_t} \end{array} \right| \cdots \left| \begin{array}{c} \frac{\lambda_{k-1}\tau_k\alpha_1}{\lambda_{k-1}\tau_k\alpha_2} \\ \vdots \\ \frac{\lambda_{k-1}\tau_k\alpha_t}{\mu_{k-1}\alpha_t} \end{array} \right|$$

For  $k$  even,

$$\mu_k(\alpha) = \mu_k \left| \alpha_1 \right| \cdots \left| \alpha_t \right| := \frac{\left| \begin{array}{c} \mu_{k-1}\alpha_1 \\ \nu_{k-1}\tau_k\alpha_1 \end{array} \right| \left| \begin{array}{c} \lambda_{k-1}\sigma_k\alpha_2 \\ \mu_{k-1}\alpha_2 \end{array} \right| \cdots \left| \begin{array}{c} \lambda_{k-1}\sigma_k\alpha_t \\ \lambda_{k-1}\sigma_k\alpha_t \end{array} \right|}{\left| \begin{array}{c} \vdots \\ \nu_{k-1}\tau_k\alpha_1 \end{array} \right| \left| \begin{array}{c} \nu_{k-1}\tau_k\alpha_2 \\ \nu_{k-1}\tau_k\alpha_2 \end{array} \right| \cdots \left| \begin{array}{c} \mu_{k-1}\alpha_t \\ \mu_{k-1}\alpha_t \end{array} \right|}$$

Having defined  $\sigma_k$  and  $\tau_k$  for  $\mathcal{I}^n$ , it is possible to define a structure on the infinite dimensional cube, characterised as

$$\mathcal{I}^\infty := \{f : \mathbb{N} \rightarrow \mathcal{I} \mid f^{-1}(0) \text{ is finite}\}.$$

Notions of well-formedness of subsets of  $\mathcal{I}^\infty$  can be given as in [9], using the cocycle conditions, and it is straightforward to prove

**Proposition 6.1:** There is an  $n$ -category structure defined on  $\mathcal{I}^\infty$ .

**Proposition 6.2:** (i) If  $\alpha$  is a full  $j$ -sub-block of  $\sigma_k[n]$ , then  $\sigma_i(\alpha) = \sigma_i[n]$  for  $i < j < k$ , and similarly for targets.

(ii) If  $\alpha *_j \beta$  is an occurrence of consecutive  $j$ -blocks of  $\sigma_k[n]$ , then  $\sigma_j(\beta) = \tau_j(\alpha)$ . Moreover, topologically the union of sub-cubes of  $\sigma_j(\beta)$  is a disk of dimension  $j$ .

(iii) The piles generated by applications of elements of the cocycle blocks are nested embeddings of  $k$ -disks  $V_k$  with geometric boundary  $\partial V_k = \sigma_{k-1} \cup \tau_{k-1}$ .

## 7 Low dimensional examples

We gave diagrams indicating how the first three cocycles arise, repeating their description in block form accompanied by the geometric decomposition corresponding to the block. Note that the 2-cocycle condition can be thought of as saying “replace the interior of the hexagon on the left by the configuration on the right”. In other words, flip from the front faces to the back ones in the standard Necker cube.

Figure 15 shows the boundary of the 4-dimensional cube, labelled by our convention. The 2-dimensional target faces are shaded, the source faces labelled, and the heavy line depicts the 1-source. To depict the 3-cocycle condition, we present the five stages of deformation of the 2-face cubes across the four 3-dimensional cubes of the 3-dimensional source faces. This corresponds to looking at the set of  $\mathcal{I}^2$ s arising in the respective piles. These pictures decompose as indicated in Figure 16, where the order is that determined by the corresponding block.

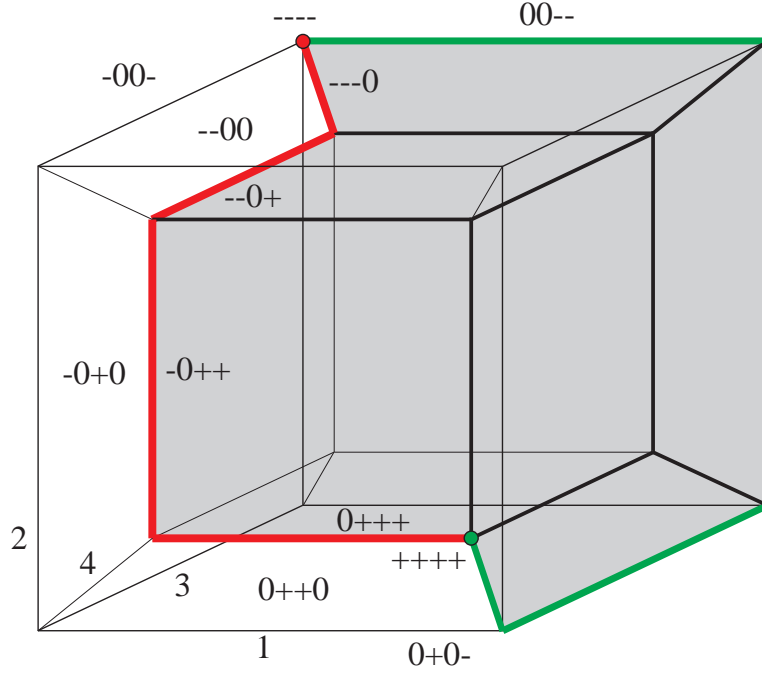


Figure 15: The shaded squares give  $\tau_2[4]$ : Left edges of the shaded squares give  $\sigma_1[4]$ ; the right edges give  $\tau_1[4]$ . Labels reflect the structure arising from [3].

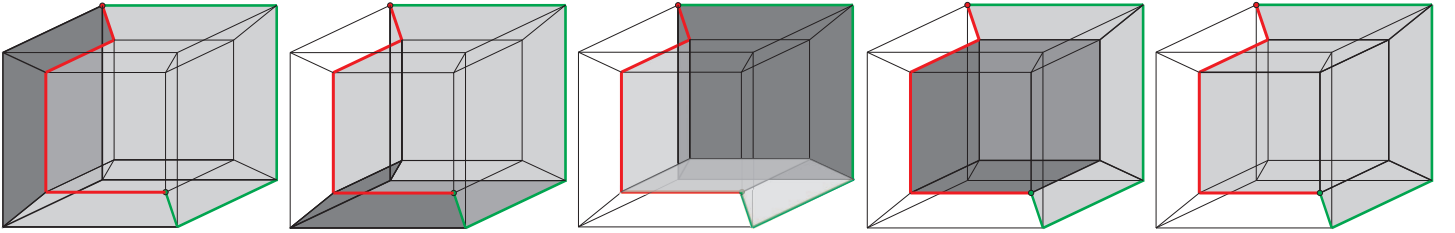


Figure 16: The action of  $\sigma_3[4]$  on  $\sigma_2[4]$ , producing  $\tau_2[4]$ .

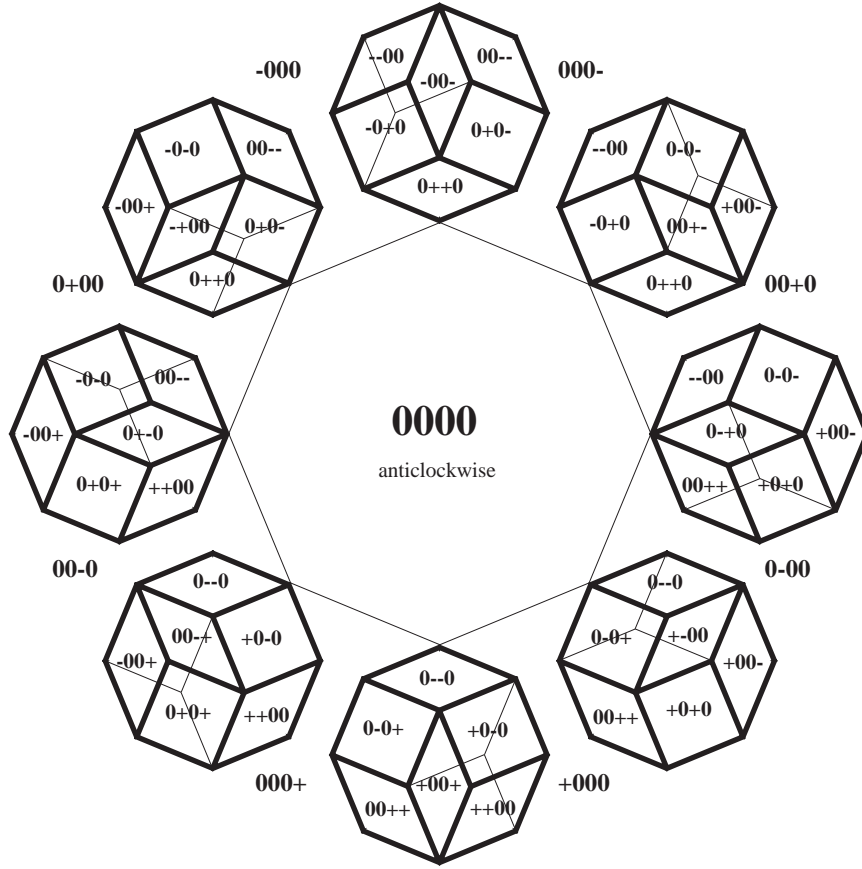
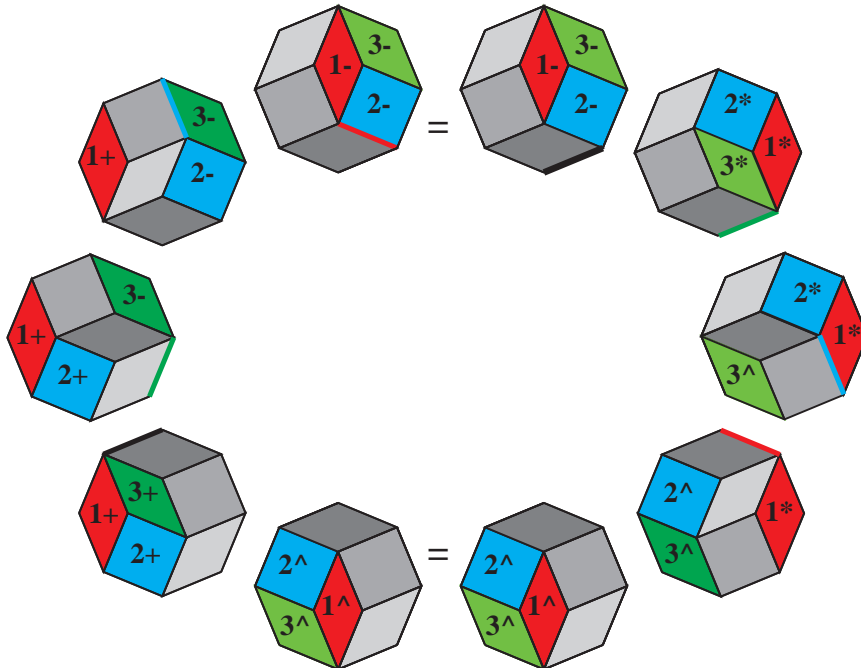


Figure 17: The Octagon of Octagons: 3-cubes in [4] acting on oriented 2-cells in the 4-cube. Observe the  $\pi$ -rotational symmetry and interchange of  $\pm$ . Faintly shaded 3-cubes are labeled correctly when read anticlockwise. Each octagon is rotated by  $3\pi/4$  at each stage, with a consequent (not described here) action on the labelings of squares.



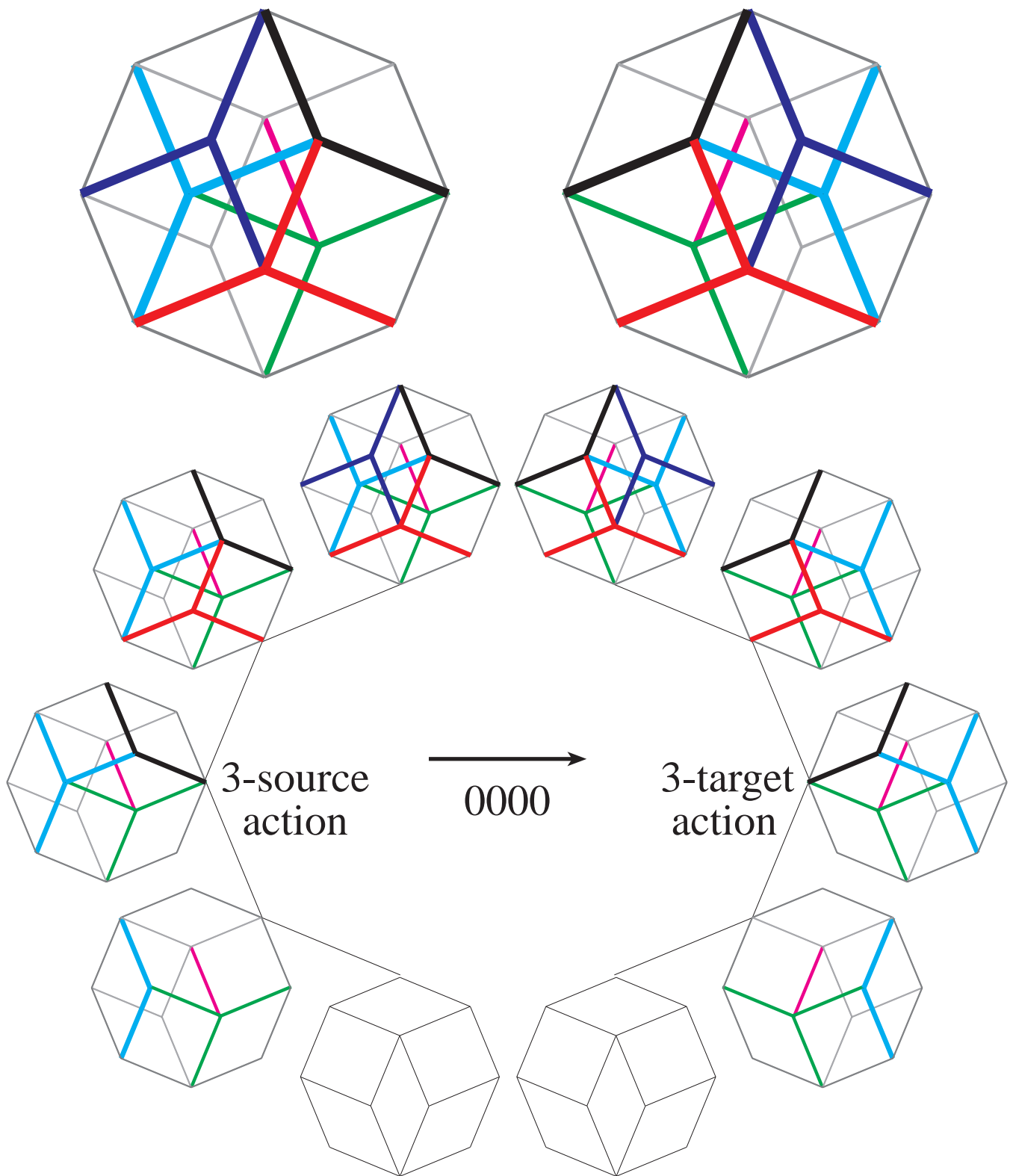
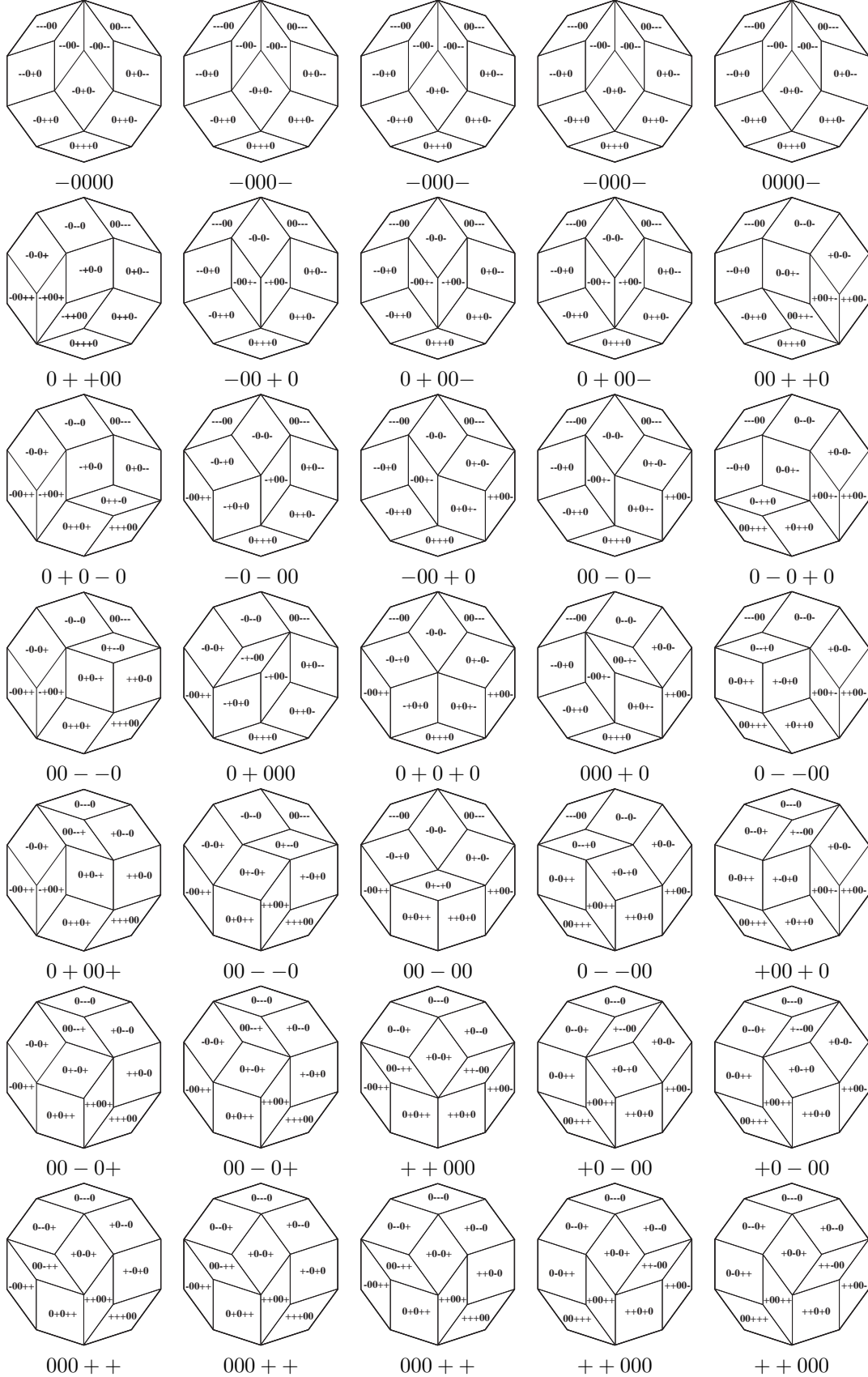


Figure 18: 3-cubes in the source and target of [4]

Figure 19: 5-cube source: and target next page



# 5-cube target

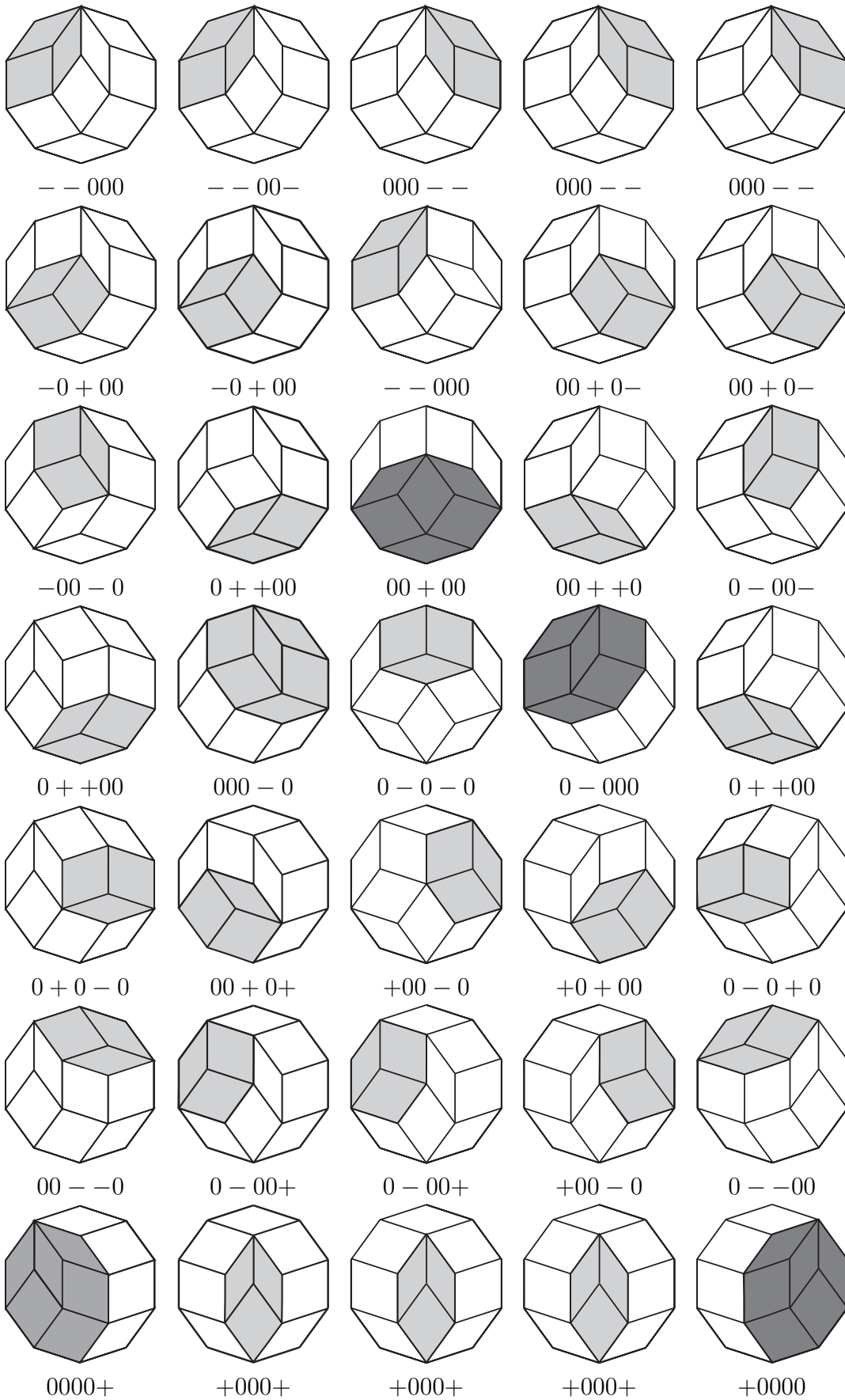


Figure 1 displays 35 octahedra, each representing a different partition of the root system of  $E_6$ . The octahedra are arranged in a 7x5 grid. Each octahedron is labeled with a symbol and its corresponding partition. The symbols are arranged in a grid: Row 1:  $\mu_{11}$ ,  $\lambda_{11}$ ,  $\lambda_{11}$ ,  $\lambda_{11}$ ,  $\lambda$ ; Row 2:  $\mu_{12}$ ,  $\mu_{21}$ ,  $\lambda_{23}$ ,  $\lambda_{23}$ ,  $\mu_1$ ; Row 3:  $\mu_{13}$ ,  $\mu_{22}$ ,  $\mu_{31}$ ,  $\lambda_{33}$ ,  $\mu_2$ ; Row 4:  $\mu_{14}$ ,  $\mu_{23}$ ,  $\mu_{32}$ ,  $\mu_{41}$ ,  $\mu_3$ ; Row 5:  $\nu_{23}$ ,  $\mu_{24}$ ,  $\mu_{33}$ ,  $\mu_{42}$ ,  $\mu_4$ ; Row 6:  $\nu_{33}$ ,  $\nu_{33}$ ,  $\mu_{34}$ ,  $\mu_{43}$ ,  $\mu_5$ ; Row 7:  $\nu_{41}$ ,  $\nu_{41}$ ,  $\nu_{41}$ ,  $\mu_{44}$ ,  $\mu_6$ . Each octahedron is a 3D representation with faces colored red, grey, or white, and labeled with variables like  $\lambda$ ,  $\mu$ ,  $\nu$ ,  $\tau$  and their indices.

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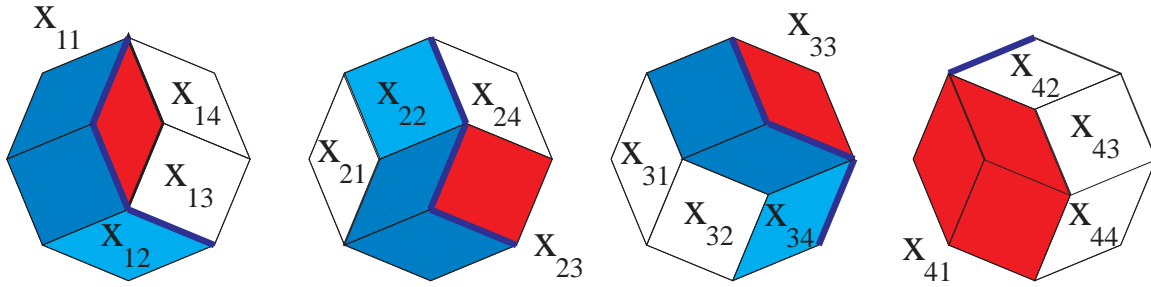


Figure 21: Labeling of the 4-cube, used for the 5-cube, arising from the 3-cube. The first of the pair of subscript labels refers to the diagram number; the second to the order in which deformations must be applied to 2-cubes/3-cubes. The subscripts for the 5-cube have been written in reverse order.

antipodal symmetry. If all symbols were to be deleted, the octagon of octagons shows that each figure, going clockwise around the octagon, differs from the preceding by a  $3\pi/4$  rotation.

In Figure 17 we show how the 3-cocycle arises from the 2-cocycle by ordering the thickenings and shiftings of cubes that occur. The left hand side of the octagon arises by considering the order of 2-cell composition for the 2-cocycle source, indicated by the numbers, and consecutively moving these across the octagon via their thickenings into 3-cubes as indicated by the bold-face hexagons. The target side of the octagon of octagons can be obtained either by symmetry considerations or by the same process.

It is clear that the unlabelled configurations of the octagon of octagons can be derived entirely without labelling *any* of the faces! Thus the only purpose of the labellings of the cube is to allow us to write down the cocycle conditions in a linear symbolic form, without the need for these diagrams.

We may replace the octagon by Figure 18, which depicts the splitting of the eight  $\mathcal{I}^3$ s in the boundary of  $\mathcal{I}^4$  into the source and target sets. The octagonal “outline” we see is the union of the 1-source and 1-target, splitting the boundary 2-sphere, which itself is the union of the 2-source and 2-target, into these two hemispheres. The 3-source of  $\mathcal{I}^4$  is on the left, the 3-target on the right. The front faces are the 2-source  $\sigma_2[4]$ , the back face  $\tau_2[4]$ . To find the upper 4 octagons of the octagon of octagons, consecutively “puncture” the  $\mathcal{I}^3$ s having three 2-faces exposed, thinking of the configuration as cubical soap bubbles. The order to proceed is uniquely determined at this dimensional level, and the collection of  $\mathcal{I}^2$ s we see at each stage gives the configurations of the octagon. We illustrate the puncturing procedure by redrawing the “octagon of octagons” as a sequence of 3-dimensional configurations of cubes.

Although it is possible to draw the “source” of the 4-cell in a single configuration – that of four cubes glued together as half of a tesseract – by doing so it is not clear how to write down a “canonical” ordering of the 2- and 3-cubes at each stage of the octagon, as a path of deformations of the 1-source to the 1-target.

The 5-dimensional cube can be treated in the same manner. It is easiest to present the data as pictures of consecutive  $v_2$ -collections arising in the sequence of piles. At the top and bottom of each column is  $\sigma_2[5]$  and  $\tau_2[5]$ , which are again mirror images. We obtain 5 columns, corresponding to the five  $\mathcal{I}^4$ s of the source (Figure 19), and another 5 columns corresponding to the target. Note the bilateral symmetry of the decagonal configurations for each of  $\sigma_4[5]$  and  $\tau_4[5]$ , and that the configurations of  $\tau_4[5]$  are those of the source read backwards with  $+$  and  $-$  interchanged. Each column of decagons provides a path from  $\sigma_2[5]$  to  $\tau_2[5]$ .

Between the decagons of any column we have written the word  $x$  whose application  $\pi_x$  transforms the decagons one to the next. Hence the shaded subsets of Figure 19 indicate  $\sigma_2(x)$  for each subsequent  $x$ . We merely replace the interior configuration by its mirror image, with the symbols behaving accordingly.

Each decagon itself gives a path from  $\sigma_1[5]$  to  $\tau_1[5]$ . Together the five columns of either  $\sigma_4[5]$  or  $\tau_4[5]$  give a path from  $\sigma_3[5]$  to  $\tau_3[5]$ . In each column occurs one  $\mathcal{I}^4$  and six  $\mathcal{I}^3$ s. These latter  $\mathcal{I}^3$ s, together with the four of the source of  $\mathcal{I}^4$ , make up the ten  $\mathcal{I}^3$ s of the pile at that stage. The shaded  $\mathcal{I}^2$ s together with those

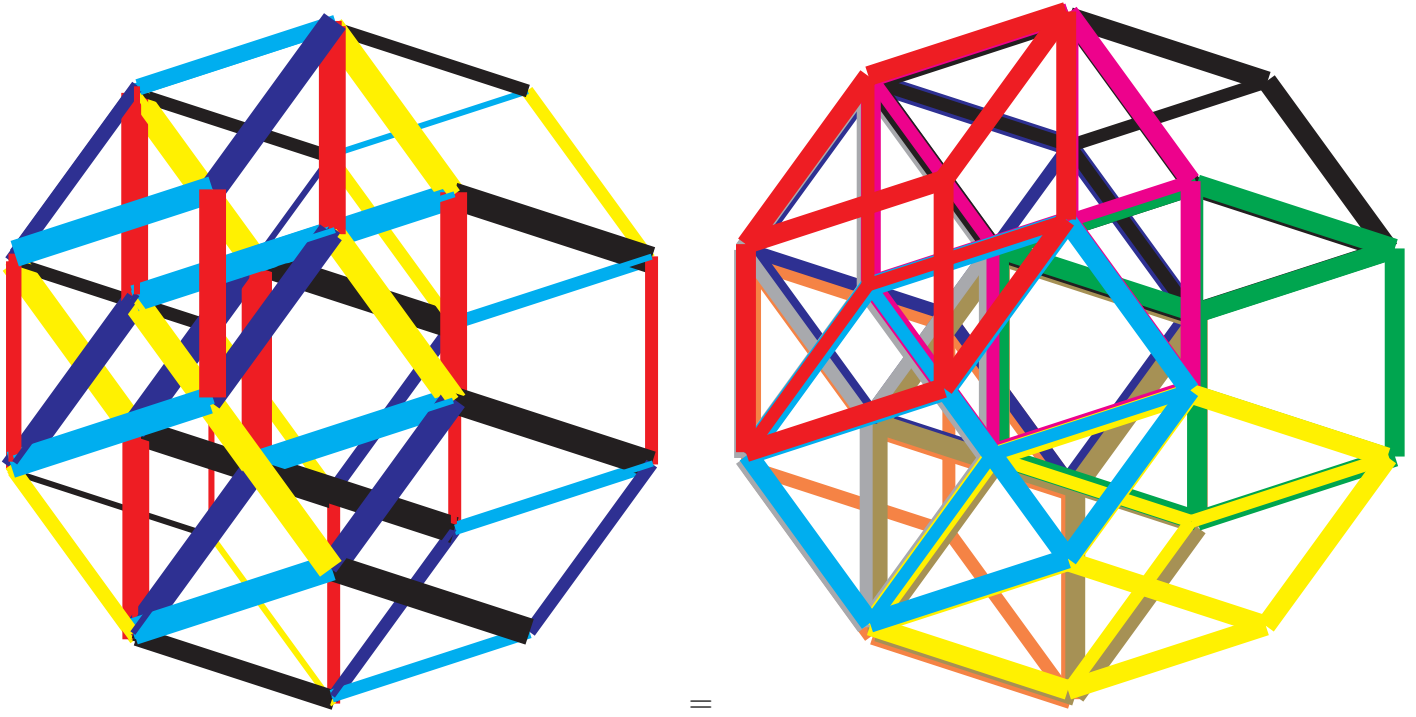


Figure 22: The 5-cube,  $\psi_3[5]$ . Left: Five edge-types, coloured. Right: ten 3-cubes, coloured. We present the interior structure of the 5-cube 3-source 3-cell in both ways for the benefit of the reader: the 10 colours refer to the number of ways three distinct edge colours can be chosen from five. Parallel edges are labeled from 1 to 5, clockwise from the top of the figure. Thus ‘145’ identifies the 3-cube at the bottom right (coloured yellow). The 0-source and target are respectively top and bottom; the 1-source and target are left and right in each decagon.

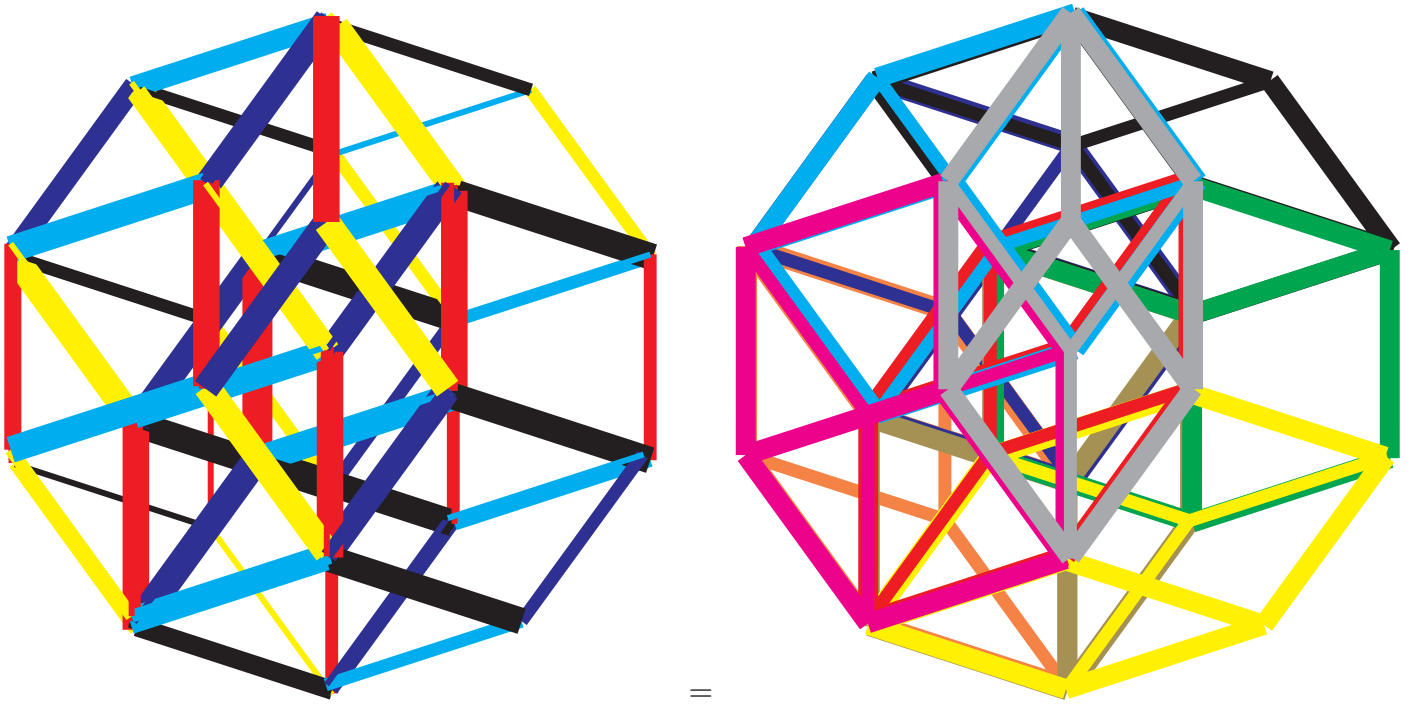


Figure 23:  $-0000(\psi_3[5])$ : front cells  $\psi_2[5]$ , rear cells  $\omega_2[5]$

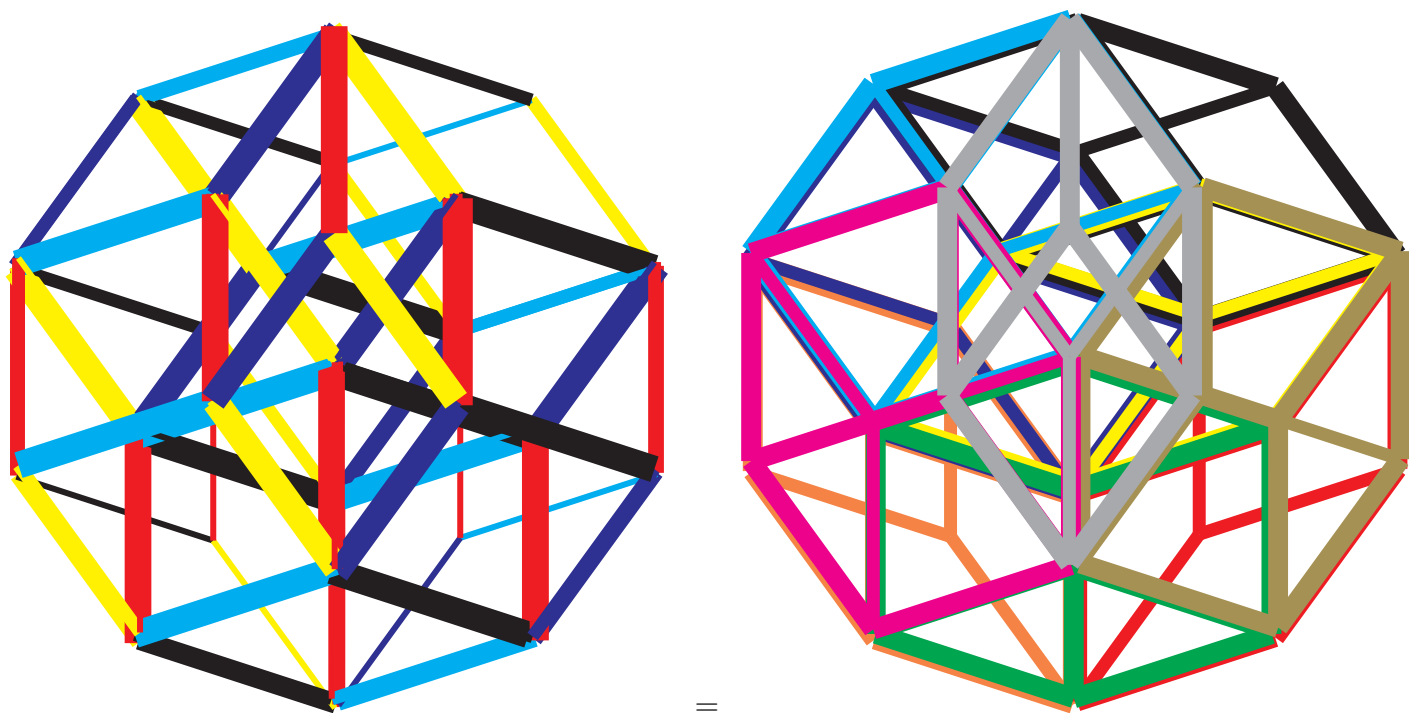


Figure 24:  $0+000[-0000(\psi_3[5])]$

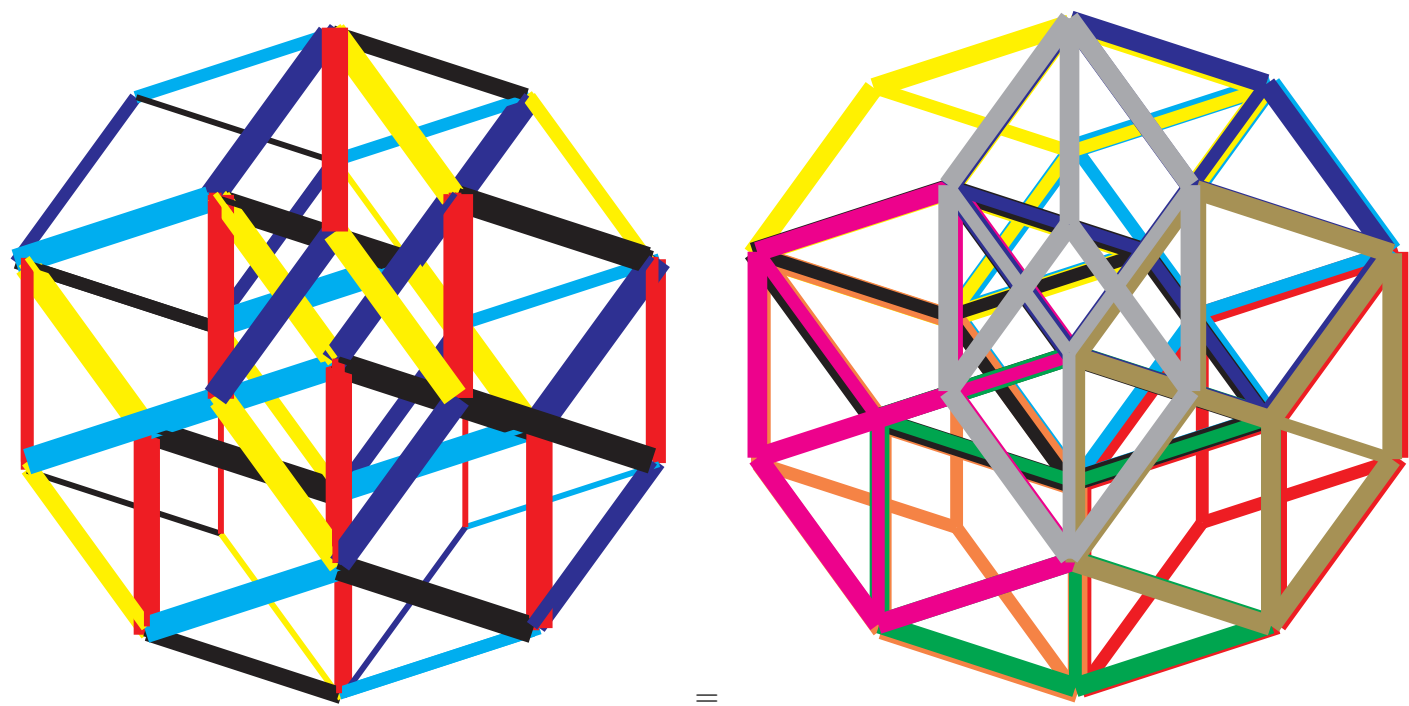


Figure 25:  $00-00(0+000[-0000(\psi_3[5])])$

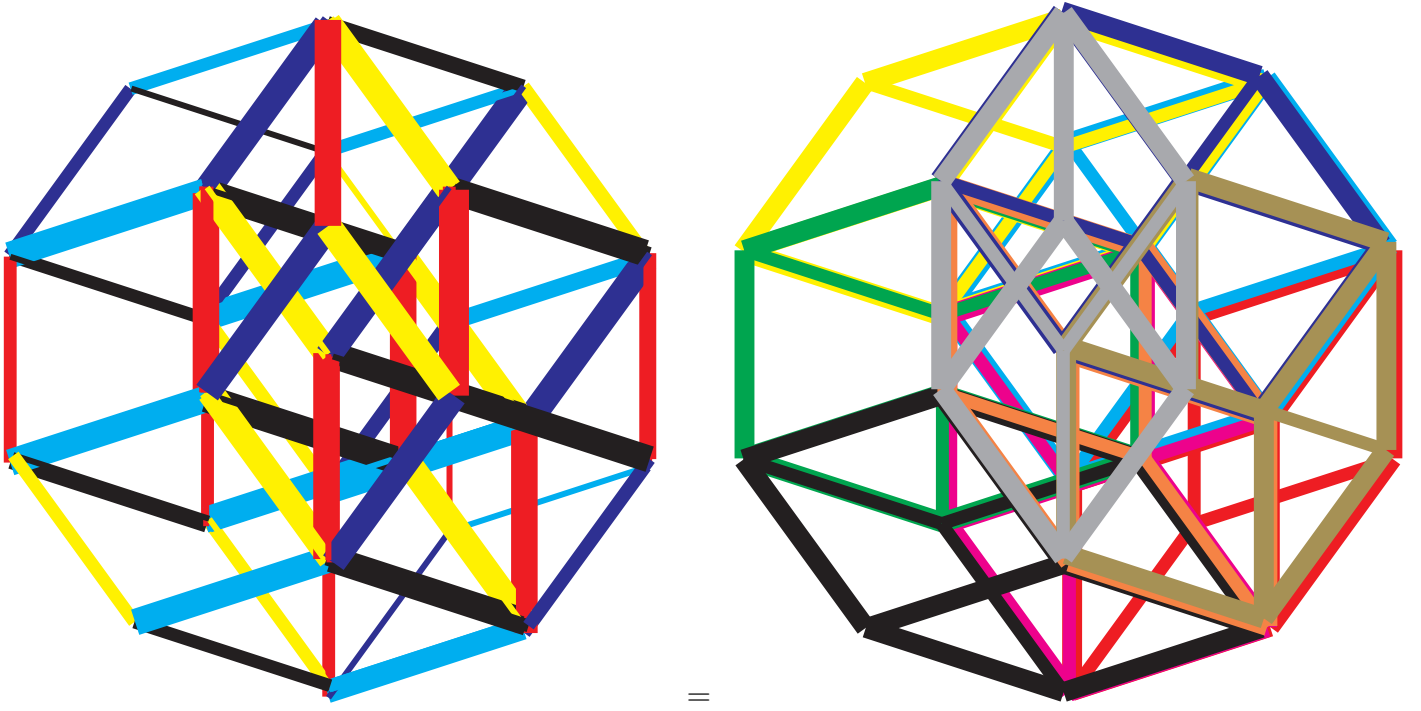


Figure 26:  $000+0[00-00(0+000[-0000(\psi_3[5]))]$

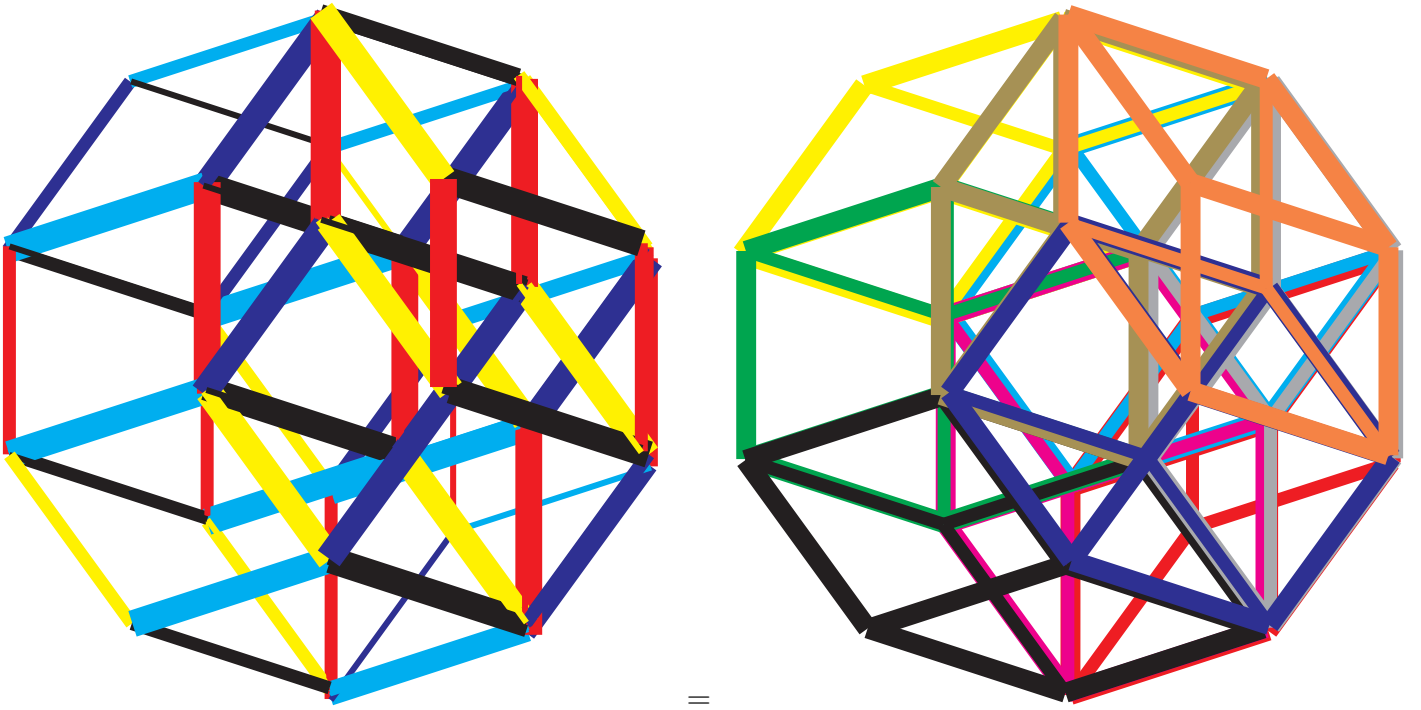


Figure 27:  $0000-(000+0[00-00(0+000[-0000(\psi_3[5]))]) = \omega_3[5]$ .

remaining make up the ten of  $v_2$  at that pile stage, accordingly giving the block structure at the 2-dimensional level, if we ignore considerations of order. From these figures we can write down a sequence of piles with their 1-dimensional sets either present or deleted.

In Figure 21, we indicate how the first four columns of  $\sigma_4[5]$  arise from  $\sigma_3[4]$ . We use the symbols  $\mu$ ,  $\nu$ ,  $\lambda$  and hatching to bring out the sub-cubes of relevance. The dotted squares correspond to the sixteen  $x_{ij}$  terms from the 4-cube source, depicted at the bottom. These exist in the 5-cube source as either old, new or thickened versions. The lined squares indicate thickenings. Note that one of the 4-dimensional configurations arises if we collapse all lined squares of a given decagon along the direction of the lines; each decagon arises by splitting an octagon along a path from the 0-source to the 0-target and inserting a square for each segment of the path (the lined squares). Observe that each column corresponds to doing some of the 4-dimensional operations in the “old” ( $\lambda$ ) copy, then moving each of the four parts of an octagon across to the “new” copy, then completing the octagon operations in the “new” ( $\nu$ ) copy. The subscripts refer to the corresponding objects in 4-dimensions, with 2-dimensional source and target faces indicated accordingly.

In the sequence of figures (Figure 22 – 27) we show how the 3-faces  $\psi_3[5]$  are glued together, and how the configuration is transformed into  $\omega_3[5]$  by a sequence of modifications of the interiors of the 3-dimensional sources of the five  $\mathcal{I}^4$ s making up  $\psi_4[5]$ . These collections of sub- $\mathcal{I}^3$ s are delineated in bold. Each copy of  $\sigma_3[4]$  is replaced by a copy of  $\tau_3[4]$ , with the same 2-dimensional boundary. As with the 2-dimensional configurations, the interiors are replaced by their mirror images, the labels changing accordingly. This enables us to write down the sequence of pile sets  $v_2$  and  $v_3$ , but neither the order nor the intermediate stages of  $v_1$ 's.

## 8 The cubical/simplicial dichotomy

We make some remarks on the relationships between the simplicial and cubical approaches. We describe how the structure on  $\mathcal{I}^n$  gives rise to structure on  $\Delta^k$ , the  $k$ -simplex, for  $k = n + 1, n, n - 1$ . We shall be more detailed for the last case, deferring the others to a subsequent paper as the questions are deeper.

### 8.1 $\Delta^{n-1}$ as a sliced corner of $\mathcal{I}^n$

The  $(n - 1)$ -simplex has vertices  $0, 1, \dots, (n - 1)$ . Each sub-simplex can be described as a word in symbols  $0$  and  $+$ , the presence of  $0$  in the  $i^{\text{th}}$  position indicating that the vertex  $i - 1$  is present in the sub-simplex. Since all such words of length  $n$  occur in our description of sub-cubes of the  $n$ -cube, we have the standard description of  $\Delta^{n-1}$  as the slice off the corner  $+\dots+$  of  $\mathcal{I}^n$  by a hyperplane.

Consider the sequence of piles occurring in the cubical cocycle condition. This gives a sequence of nested embedding of  $k$ -disks into the cube, at each dimension having fixed boundary. Each pile is a collection of sets of words from  $\mathcal{I}^n$ , some of which do not contain ‘ $-$ ’. If we restrict to the subsets within the pile consisting of those sub-cubes in whose word description no ‘ $-$ ’ occurs, we see that the action of  $\pi_x$  for  $x$  containing a ‘ $-$ ’ symbol leaves the sub-pile unaltered. Only for those  $x$  consisting entirely of  $0$ 's and  $+$ 's do we change the sub-set. Since geometrically every such  $x$  intersects the slicing hyperplane in one dimension less, we see that cubical pile modifications induce a structure on the simplex also describable by piles and blocks, but where now a word in  $0$ 's and  $+$ 's corresponds to a sub-simplex of dimension  $|x| - 1$ . In particular, if we look at the source and target  $(n - 1)$ -face cubes of  $\mathcal{I}^n$ , we pick out all  $(n - 2)$ -faces of  $\Delta^{n-1}$ . These are accordingly divided into two sets, the source and target faces. Note that the former correspond to the consecutive deletion of *odd* vertices, whereas to obtain the target faces we delete *even* vertices from  $\Delta^{n-1}$ . There is a simple procedure to derive the induced cocycle condition for  $\Delta^{n-1}$  from that of the cube  $\mathcal{I}^n$ : Rewrite according to the rules

1. Starting with  $*_{n-2}$  and proceeding down to  $*_0$ , consider the highest dimensional cube between successive  $*_k$ 's.

2. For each such pair, if the highest dimensional term occurring between them contains a ‘ $-$ ’, delete all terms between the pair, and one of the  $*_k$  terms.
3. Delete all vertices and occurrences of  $*_0$ .
4. For the words remaining, write  $i - 1$  for each occurrence of 0 in the  $i$ th position.
5. Replace each  $*_k$  by  $*_{k-1}$ .

**Examples 8.1:** For the cocycle conditions previously described, we obtain exactly the description of simplicial cocycles obtained by Street [9]. For the 1-, 2-, 3-cubes, we obtain Figure 28. In Figure 29 we do the same for the 4-cube, the upper diagram giving the data in terms of the ‘octagon of octagons’, the lower the interpretation on the tetrahedron slice  $\Delta^3$  acting on paths.

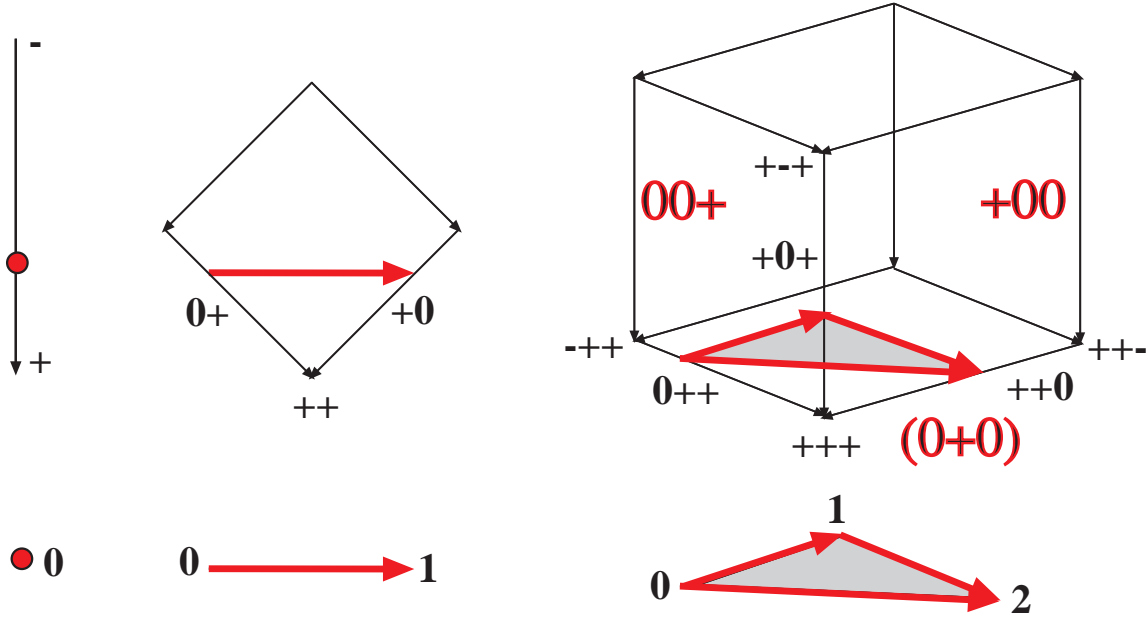


Figure 28: Slicing off a corner of cube with labeled sub-cubes gives a simplex with labeled sub-simplices

In Figure 30 we show that the 5-dimensional cube gives rise to the famous ‘pentagon of pentagons’ arising in homotopy coherence theory as well as in the definition of bicategories. We refer to [9] for further discussion. The pentagon of pentagons can be seen directly from Figures 22 – 27, by slicing the bottom of each figure with a horizontal plane. The vertices of the pentagon are determined as the intersection between the plane and the edges emanating from the 0-target of  $\mathcal{I}^5$ , the internal configuration of the triangles arising from intersections with  $\mathcal{I}^2$ s. Alternatively we can interpret the decagons as providing deformations of the 0-source of  $\Delta^4$  to the 0-target, each column corresponding to a configuration of triangles in  $\Delta^4$ . We shall shortly obtain two other derivations of the pentagon of pentagons.

In [9] the structure of an  $n$ -category is shown to exist freely on the  $n$ -simplex. This derives from Street’s notion of sources and targets in terms of even and odd faces. From his results on ‘excision of extremals’, it is possible to derive the cocycle conditions for this structure, non-canonically since order is not determined. Alternatively, the  $n$ -category structure can be defined if we have the cocycle conditions. Since at the level of  $(n - 1)$ -faces of the  $n$ -simplex we have derived the same structure as in [St], we conclude

**Theorem 8.2 ([9]):** (a)  $\Delta^n$  admits the structure of an  $n$ -category.

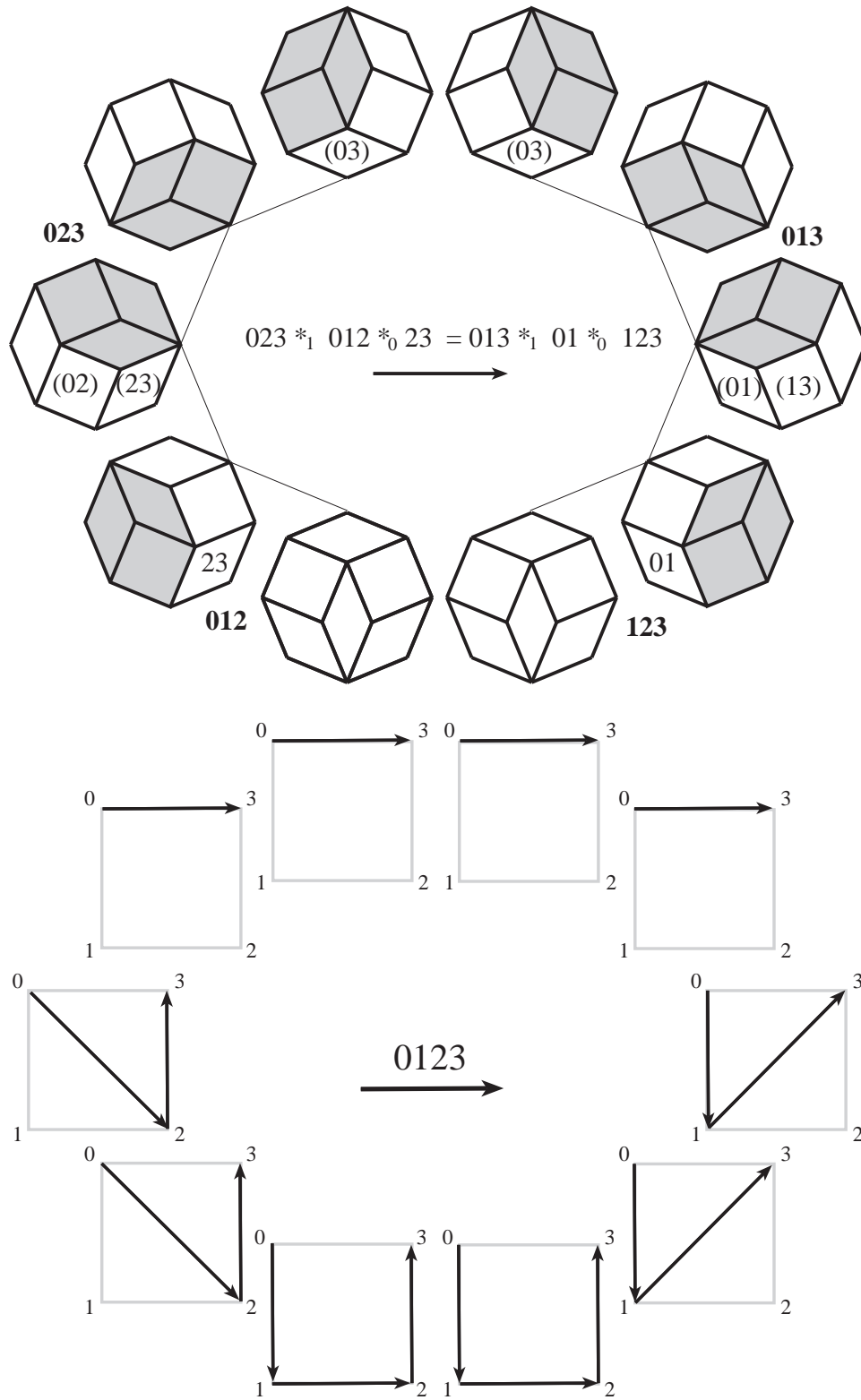


Figure 29: Deriving simplicial cocycles form cubical cocycle

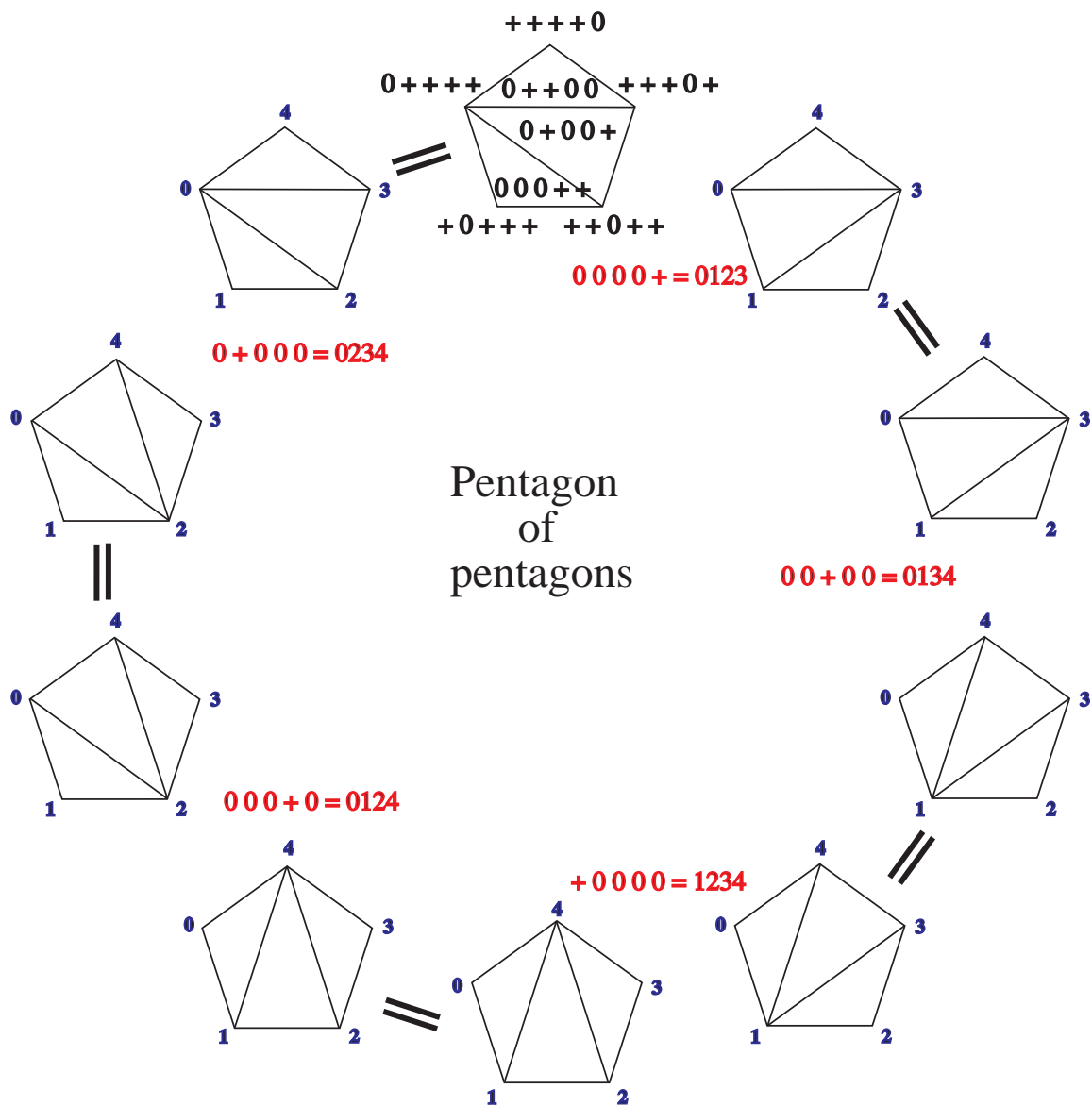


Figure 30: Slicing the 5-cube, giving the pentagon of pentagons

- (b) Street’s structure of orientals can be derived from the structure of oriented cubes, by slicing.
- (c) There is a canonical form inductively defined for the expression of simplicial cocycle conditions.
- (d) The simplicial cocycle conditions can be written in a form involving neither brackets nor the 2-categorical middle interchange law.

Roberts had conjectured that there should be a natural way of expressing the cocycle conditions without brackets.

The two approaches that follow, relating cubes and simplices, also lead to induced cocycle conditions, indicating just how ‘rigid’ and canonical is the structure.

## 8.2 Stretching a simplex into a cube

We show how the cocycle condition on  $\mathcal{I}^n$  gives rise to one on  $\Delta^n$ . To do this we briefly describe how to ‘stretch’ the  $n$ -simplex into a cube.

1. For dimension one,  $\Delta^1$  and  $\mathcal{I}^1$  coincide.
2. Higher dimensional simplices arise by ‘coning’, whereas cubes arise by taking the product with  $I$ , the unit interval. We use labellings of vertices of simplices by the symbols  $0, 1, \dots, n$ . For the 1-simplex, vertices 0 and 1 (Figure 31), the 1-cell is labelled 01.

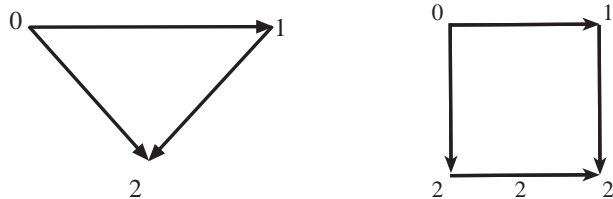


Figure 31: Expanding a labeled 2-simplex into a labeled 2-cube

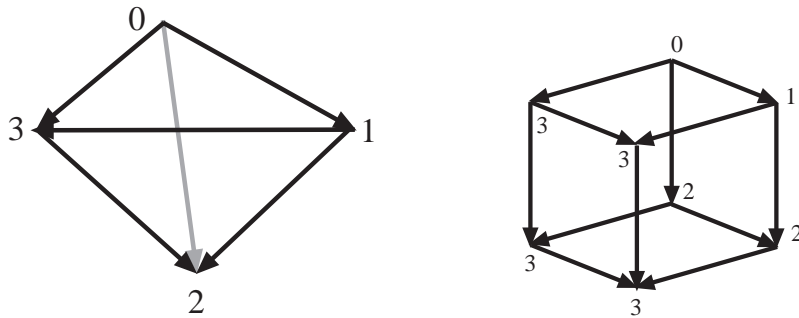


Figure 32: Expanding a labeled 3-simplex into a labeled 3-cube

Take the 2-cube whose other two vertices are labelled ‘2’. The edges are now 01, 02, 12, and ‘2’. The latter edge is really a stretched out vertex, corresponding to an ‘identity arrow’ in the categorical setting. Note that if we identify everything labelled ‘2’ together, we obtain a triangle (2-simplex).

3. Taking the product with the interval, and labelling all new vertices by ‘3’ leads to the 3-cube labelled as in Figure 32. Again, collapsing together like-labelled objects yields the tetrahedron. The source and target configurations for the 3-cube yield Figure 33, in which each 2-cell is labelled by its vertices. We obtain the simplicial 2-cocycle condition from the cubical one by direct substitution, deleting degenerate ( $\equiv$  identity) cells. Note that the only change from the source to the target configuration is the internal shape and re-labelling of the vertex.

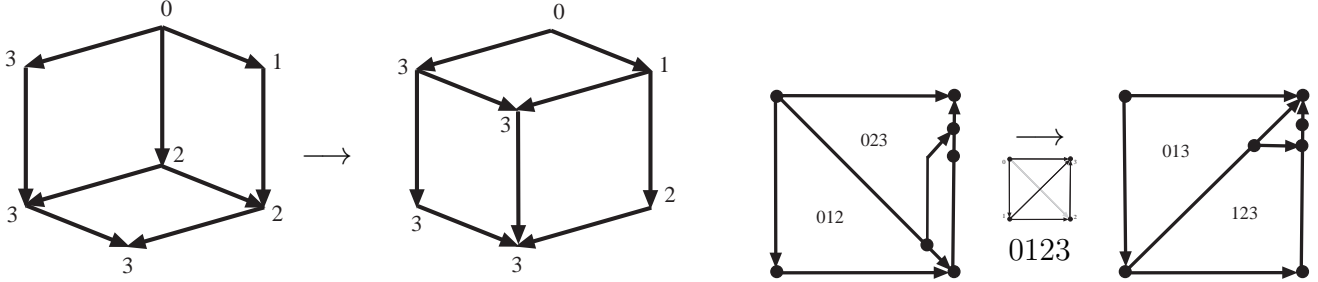


Figure 33: The 3-cubical 2-cocycle becomes the 3-simplicial 2-cocycle

4. Proceed in the same way to obtain  $\Delta^n$  ‘exploded’ into  $\mathcal{I}^n$ .
5. The rule to reverse the process is to replace a vertex of the cube, as a word in  $-$  and  $+$ , by the position of the highest occurring  $+$ , or 0 if none occur. For an arbitrary word in  $-$ , 0 or  $+$ , we also record all positions of 0 beyond the highest  $+$ . This enables us to relabel each sub-cube as an exploded sub-simplex.
6. Now rewrite cubical cocycles by replacing words in this fashion, forgetting those (as in 1 of the slicing approach) corresponding to ‘identities’. These identities are those simplices of dimension less than the cube from which they arise in this correspondence.

In Figure 34, we show how the octagon of octagons gives rise directly to the pentagon of pentagons. The upper figure gives direct substitution, the lower a deformation of the figures into a more recognisable shape.

The cubes arising in this fashion have a number of faces which are ‘identities’. The same happens in the following procedure:

### 8.3 Strings on simplices

Any (ordered) path from 0 to  $n$  on  $\Delta^n$  is uniquely determined by listing the vertices through which it passes. Since the list for any such path contains 0 and  $n$ , we need not include these in a specification of the path. If we insist on listing vertices in increasing order of magnitude, and  $i$  and  $j$  are two consecutive integers in the list with  $j \neq i + 1$ , then for each triangle  $ikj$  with  $i < k < j$  there is a triangle in the  $n$ -simplex and a new path obtained by including  $k$ . In this way triangles act on paths, and thus label morphisms in the (discrete) space of paths.

The resulting complex, with paths as objects and triangles as morphisms (acting as deformations of one path to another), is well known to form  $\mathcal{I}^{n-1}$ . We refer the reader to [3] or [7]. Not all faces of the cube correspond to faces of the simplex, some as in the previous procedure corresponding to ‘identities’. There is fortunately a well defined procedure to see which sub-cubes correspond to sub-simplices. Some repetitions occur, as can be understood by considering the effect of a triangle on a path: different paths may about the same triangle, giving rise to distinct vertices of the cube, but with edges emanating having the same triangular

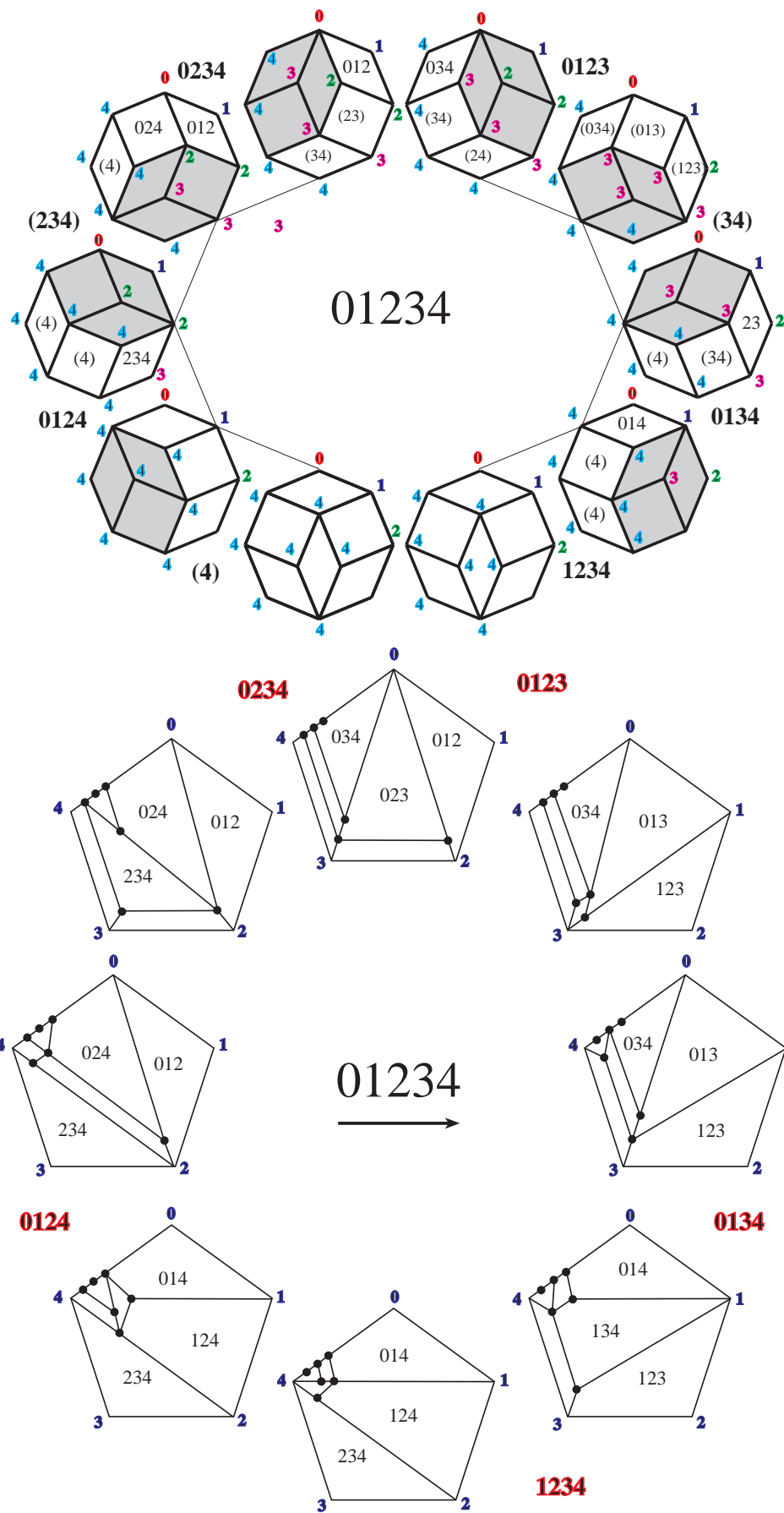


Figure 34: Deriving the pentagon of pentagons from the octagon of octagons

correspondence. The application of this approach to the understanding of simplicial oriental cocycles was initiated by Duskin [4].

Here is the prescription for converting the cubical  $(n - 1)$ -cocycle condition into the simplicial  $(n - 1)$ -cocycle condition:

1. Consider terms of dimension  $n - 1$ . A unique such occurs between consecutive  $*_{n-2}$ 's.
2. If any 0's in the word are separated by a +, delete all terms between the prior and subsequent  $*_{n-2}$ . Delete one of the  $*_{n-2}$ 's.
3. If no + occurs before a 0, write 0.
4. If a + occurs before a 0, record the highest position in which a + occurs before a 0 is encountered.
5. Record all positions in which a 0 occurs.
6. If no + is then reached before the end of the word, write  $n + 1$ .
7. If a + does occur to the right of all 0's, record the position of the first such encountered.
8. Now look at  $(n - 2)$ -dimensional words between the  $*_{n-3}$ , and follow the analogues of steps (2) and (3).
9. Repeat this process for every dimensional word.
10. Replace each  $*_i$  by  $*_{i+1}$ .
11. Insert the terms  $(0.1) * _0 (1.2) * _0 \dots * _0 (i - 1.i) * _0$  before the word, if the first number of a word is  $i$ . If the last letter is  $j$ , add after the word the symbols  $* _0 (j.j + 1) * _0 \dots * _0 (n.n + 1)$ .

**Examples 8.3:** In Figures 35 and 36 we show again how the usual simplicial cocycle conditions arise. From the 3-cube we obtain the pentagon of pentagons, from which it is also possible to see that the ‘identity’ face of the 3-cube corresponds to a *re-ordering* of the triangles across which we are deforming the 1-source of  $\Delta^4$  onto the 1-target. Hence the cube contains some additional information about structure manifest at the simplicial level.

It is possible to iterate this procedure, and consider strings on the cube. Doing so leads to the *parallelahedra*, which in dimensions 2- and 3- are the hexagon and truncated octahedron. The parallelahedra are known to tessellate Euclidean space, and like the cube and simplex have an intimate relationship with the symmetric groups. We refer the reader to Baues [3] for applications to the geometric cobar construction. Another discussion occurs in Leitch [7]. We also remark that to carry over to the parallelahedra the  $n$ -category structures discussed above should be possible in a geometric fashion, there being an inductive definition for the construction of parallelahedra. This is more difficult to describe directly as the parallelahedra are no longer regular figures. A categorical approach should exist.

These descriptions for obtaining simplicial cocycles from cubical ones show that the cubical and simplicial cocycles/structures are deeply interlocked. The procedures ‘take strings’, ‘slice’ and ‘explode’ provide a sequence of correspondences of structures: three modifications from each of the  $n$ -cubical cocycles, and three ways to obtain each of the  $(n - 1)$ -simplicial cocycles:

$$\begin{array}{ccccccccccc}
 \dots & \longrightarrow & \mathcal{I}^{n+1} & \longrightarrow & \mathcal{I}^n & \longrightarrow & \mathcal{I}^{n-1} & \longrightarrow & \mathcal{I}^{n-2} & \longrightarrow & \dots \\
 & & & & \text{string} & & \text{slice} & & \text{string} & & \\
 & & & & \swarrow & & \searrow & & \swarrow & & \\
 & & & & ex \downarrow & & ex \downarrow & & & & \\
 \dots & \longrightarrow & \Delta^{n+1} & \longrightarrow & \Delta^n & \longrightarrow & \Delta^{n-1} & \longrightarrow & \Delta^{n-2} & \longrightarrow & \dots
 \end{array}$$

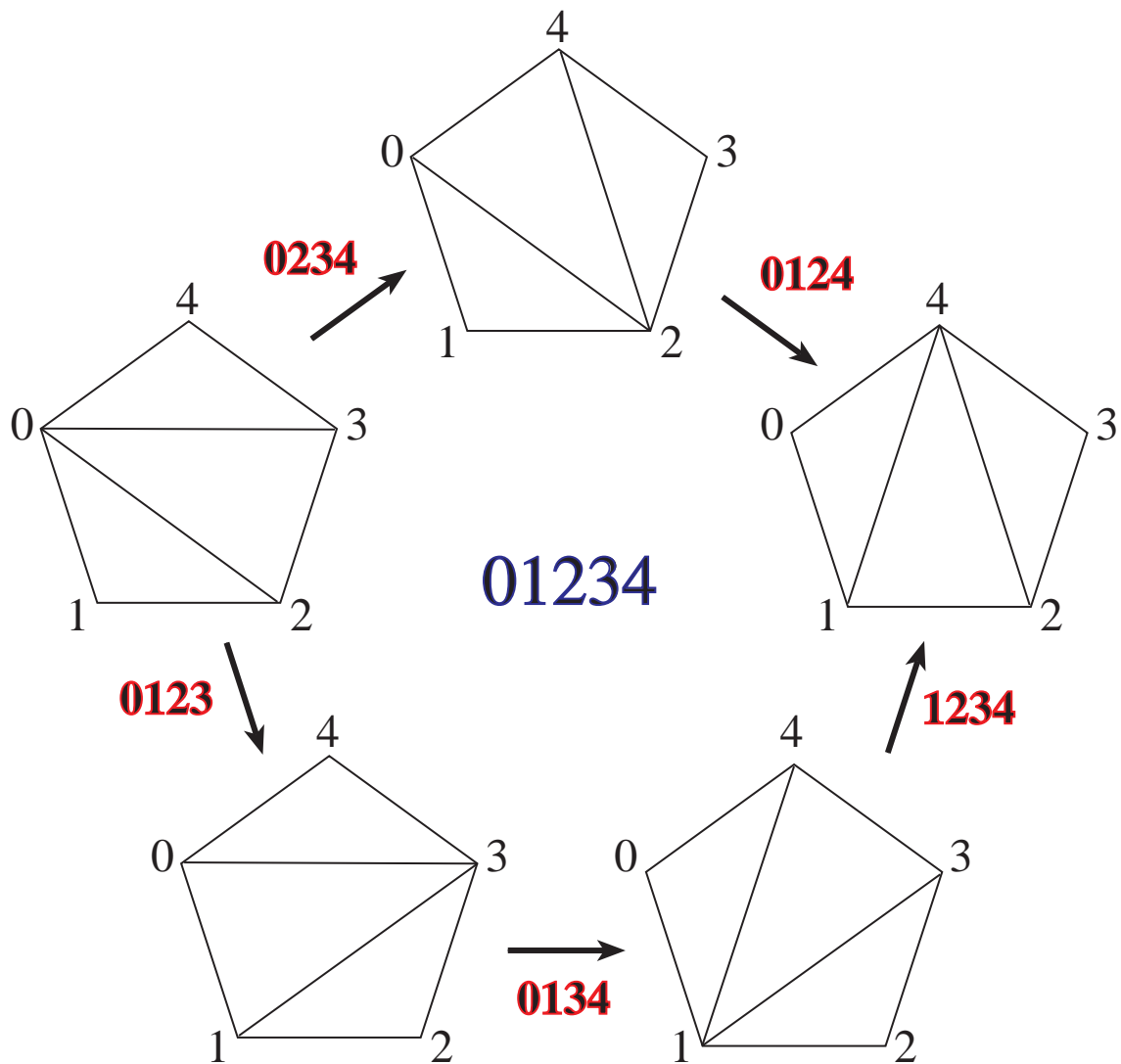
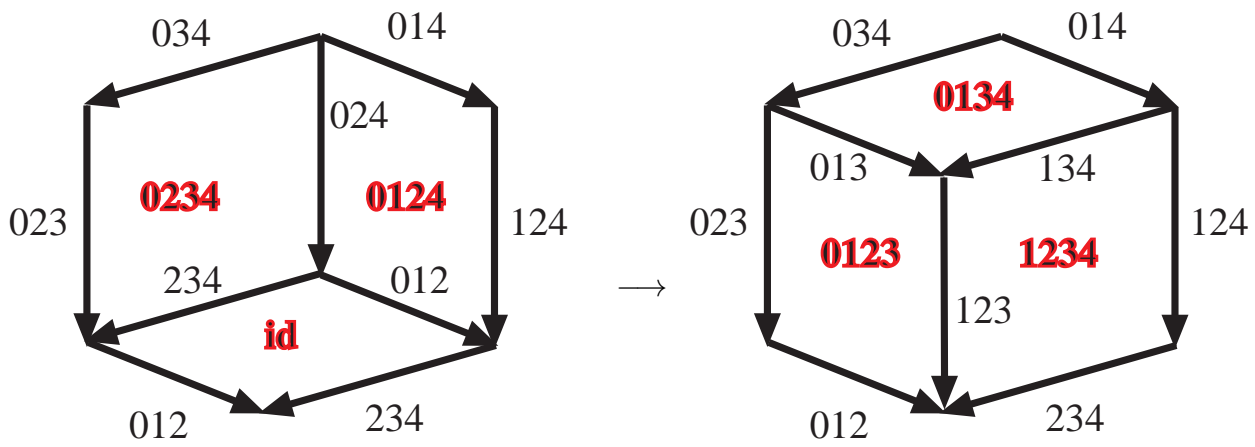


Figure 35: The pentagon of pentagons as the 3-cube ‘Yang-Baxter’ equation

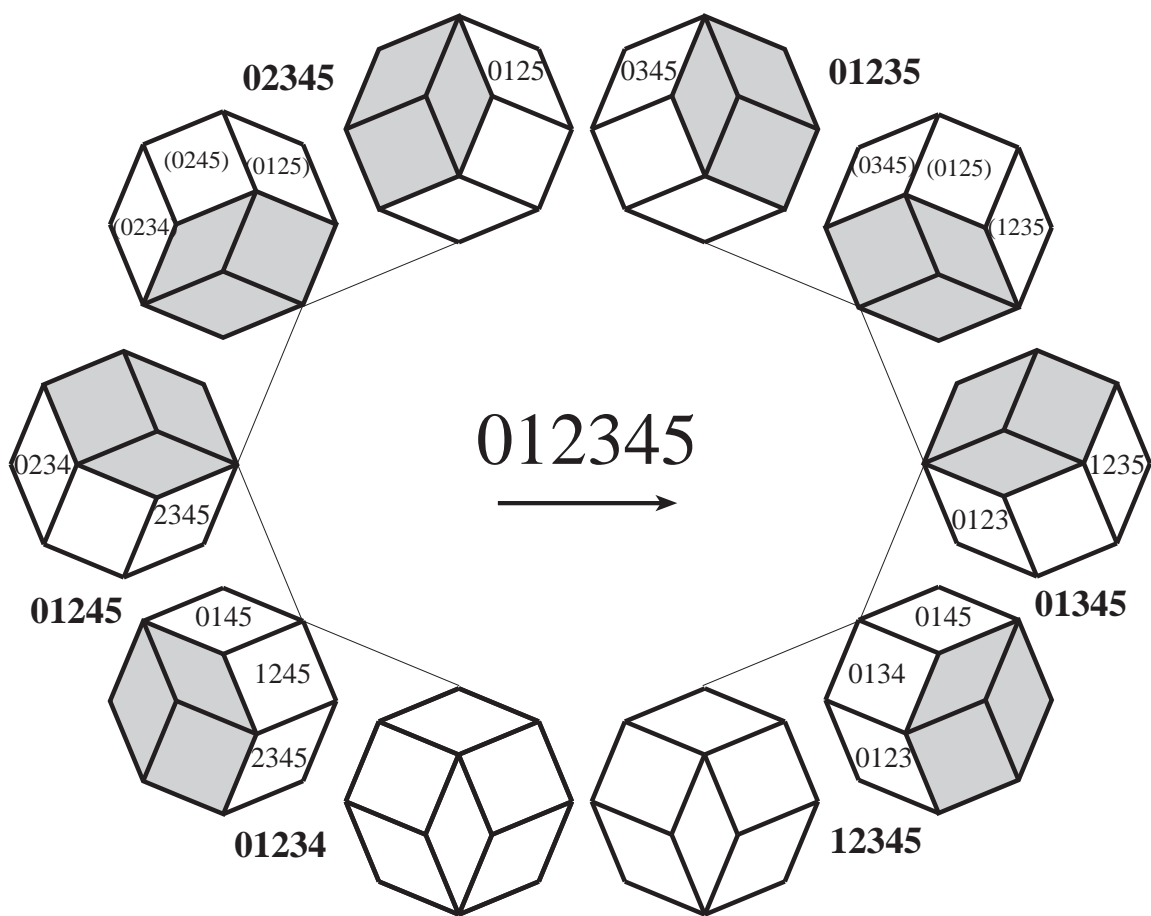


Figure 36: The 4-cube octagon of octagons giving rise to the 5-simplicial 4-cocycle

There is an analogue of the ‘ $k$ -blocks’ for cubes in the simplicial context, and a manifestation of the symmetries of the cubical cocycles. On the other hand, the three relationships described above show how the structure on the cubes gives rise to a structure on simplices, but it is not clear how the procedure can be reversed for *any* of the approaches!

It would be of interest to investigate further the commutativity of the above sequences, with respect to face and degeneracy maps for cubes or simplices. In fact, to put the whole structure in a more explicitly categorical framework would be desirable. The symmetries of many of the objects also suggests possible extensions of Connes’ cyclic homology, for which we refer to Goodwillie [5] for a categorical approach.

### Added 2009: 3-cell recombination for the 5-cube [5], figures after references

The 3-source of the 5-cube [5] contains  $10 = \binom{5}{3}$  3-cubes. Each cube is a parallel class determined by the location of three 0’s in a string of length 5, there being  $2^2 = 4$  possibilities in total corresponding to the remaining  $\pm$  entries. An application of a 4-cube in the 5-cube 4-source may change the representative: each 3-cube class is involved twice in changing the two  $\pm$  entries to  $\mp$  corresponding to the 3-cubes in the 3-target, which are the first and last columnns. We only record the position of 0’s in intermediate columnns.

Each column bold indicates source 3-cubes of the operational 4-cube: Observe that in each case reordering can be achieved before application, so that these source 3-cubes occur consecutively. Partial order rearrangement is possible when two 3-cubes do not share a common face (pair of numbers): two cubes intersect on at most 1 vertex. (This is clearer in the dual Pascal’s triangle version). The columns before the double vertical lines, read from bottom to top, enable the filling of the 3-ball by 10 distinct 3-cubes.

– – 000	<b>345</b>	<b>345</b>	234	234	234	234	234	234	<b>234</b>	<b>234</b>	123	123	000 – –
–0 + 00	<b>245</b>	<b>245</b>	235	235	235	235	<b>235</b>	134	<b>134</b>	<b>134</b>	124	124	00 + 0–
0 + +00	145	<b>235</b>	245	245	<b>245</b>	134	134	124	<b>124</b>	<b>124</b>	134	134	0 – 00–
–00 – 0	<b>235</b>	<b>234</b>	<b>345</b>	<b>345</b>	134	135	<b>135</b>	<b>235</b>	<b>123</b>	<b>123</b>	234	234	+000–
0 + 0 – 0	135	145	<b>145</b>	<b>145</b>	135	<b>245</b>	124	<b>135</b>	125	125	125	125	00 + +0
00 – –0	125	135	<b>135</b>	<b>135</b>	<b>145</b>	<b>145</b>	<b>125</b>	<b>125</b>	135	135	135	135	0 – 0 + 0
–000+	<b>234</b>	125	125	<b>134</b>	345	<b>125</b>	145	<b>123</b>	235	235	235	235	+00 + 0
0 + 00+	134	134	<b>134</b>	125	<b>125</b>	<b>124</b>	245	145	145	145	145	145	0 – –00
00 – 0+	124	124	124	124	<b>124</b>	345	345	245	245	245	245	245	+0 – 00
000 + +	123	123	123	123	123	123	<b>123</b>	345	345	345	345	345	+ + 000

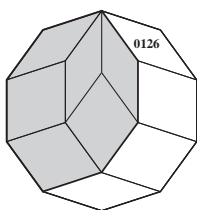
The following figures, from the original, indicate how the simplicial cocycle  $\mathcal{O}_6$  from the 6-simplex derives from the 5-cube: three nontrivial columns occur in the 5-cube source, and four in the target. Observe from the first 5-cube source column the four- and five-simplices, or considering only source four-simplices:

$$< 023456 > * < 02356 > * < 01245 > * < 01234 > \implies < 03456 > * < 02356 > * < 02345 > * < 01256 > * < 01245 > * < 01234 >$$

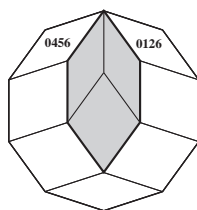
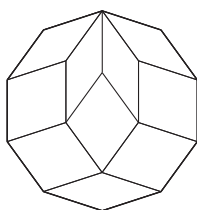
which are the labeled 2-cells in  $F_{16}$  of Street [9]. Adding in the cubically-ordered listed three-simplices gives

$$\begin{aligned} &< 023456 > * < 0126 > \\ & ** < 0234 > * < 0245 > * < 2345 > * < 01256 > * < 2356 > * < 3456 > \\ & ** < 0234 > * < 01245 > * < 2345 > * < 0156 > * < 1256 > * < 2356 > * < 3456 > \\ & ** < 01234 > * < 0145 > * < 1245 > * < 2345 > * < 0156 > * < 1256 > * < 2356 > * < 3456 > \end{aligned}$$

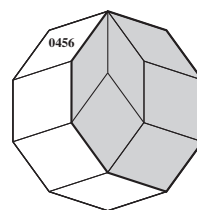
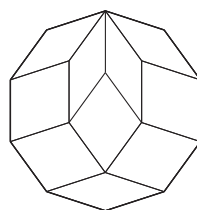
$$\begin{aligned} = & (< 03456 > * < 0236 > ** < 0345 > * < 02356 > * < 3456 > ** < 02345 > * < 0256 > * < 2356 > * < 3456 >) * < 0126 > \\ & ** < 0245 > * < 2345 > * < 02356 > * < 01256 > * < 2356 > * < 3456 > \\ & ** < 0234 > * < 01245 > * < 2345 > * < 0156 > * < 1256 > * < 2356 > * < 3456 > \\ & ** < 01234 > * < 0145 > * < 1245 > * < 2345 > * < 0156 > * < 1256 > * < 2356 > * < 3456 > \end{aligned}$$



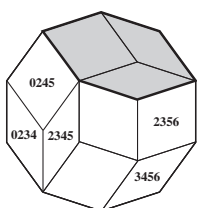
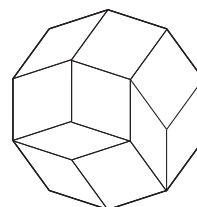
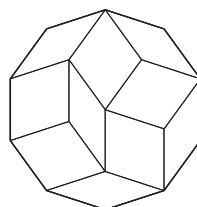
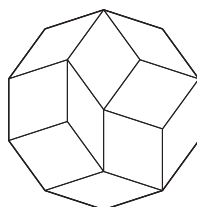
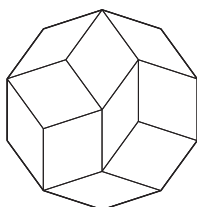
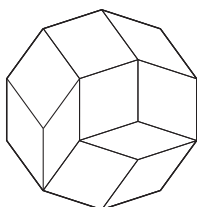
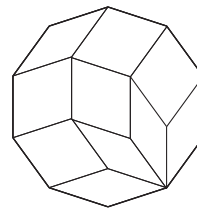
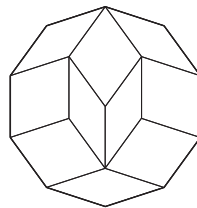
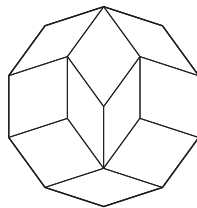
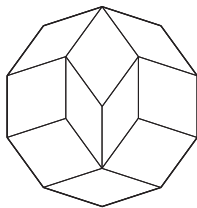
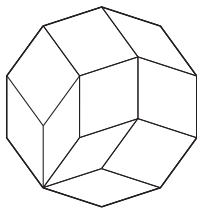
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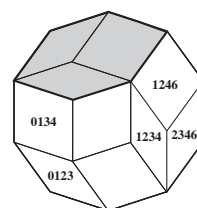
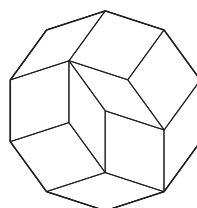
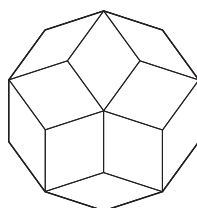
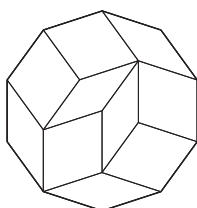
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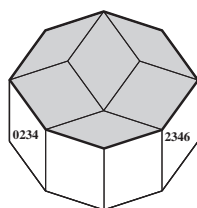
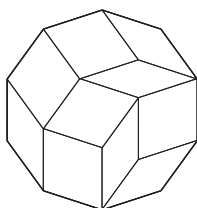
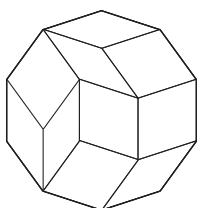
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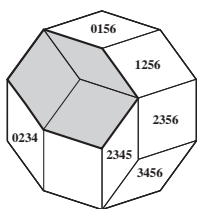
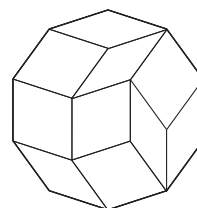
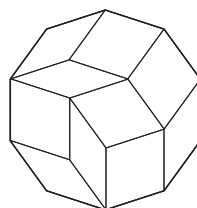
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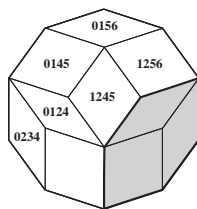
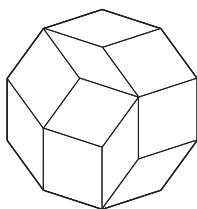
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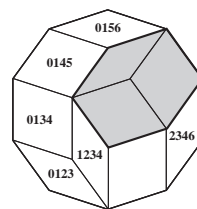
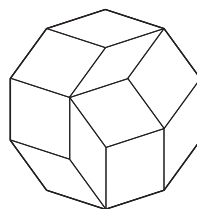
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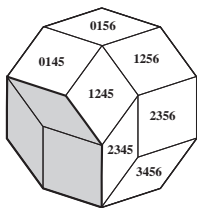
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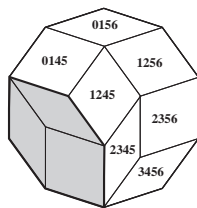
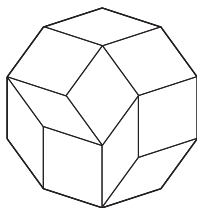
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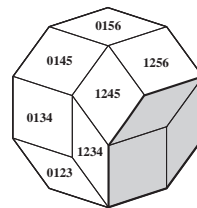
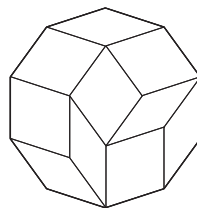
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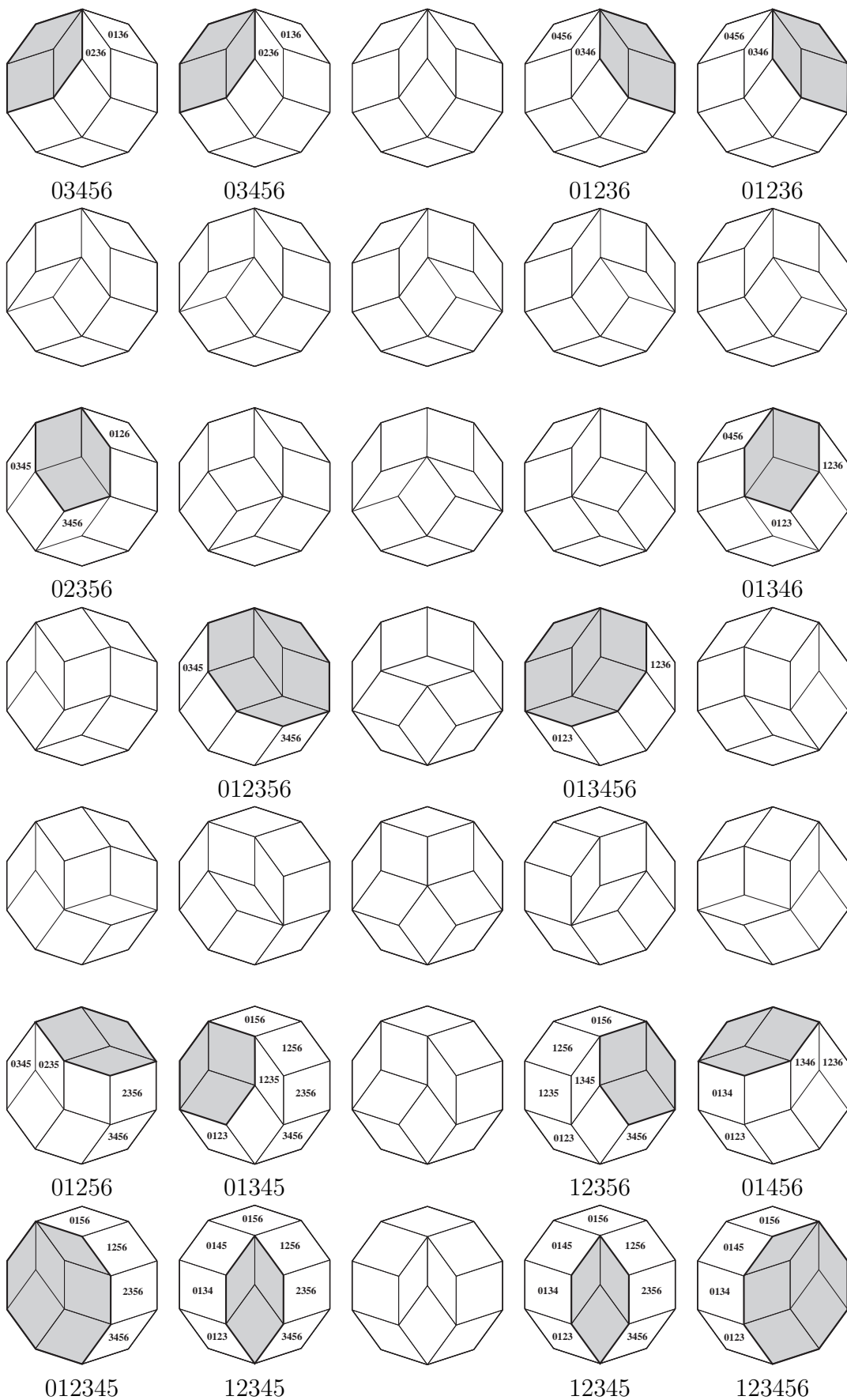
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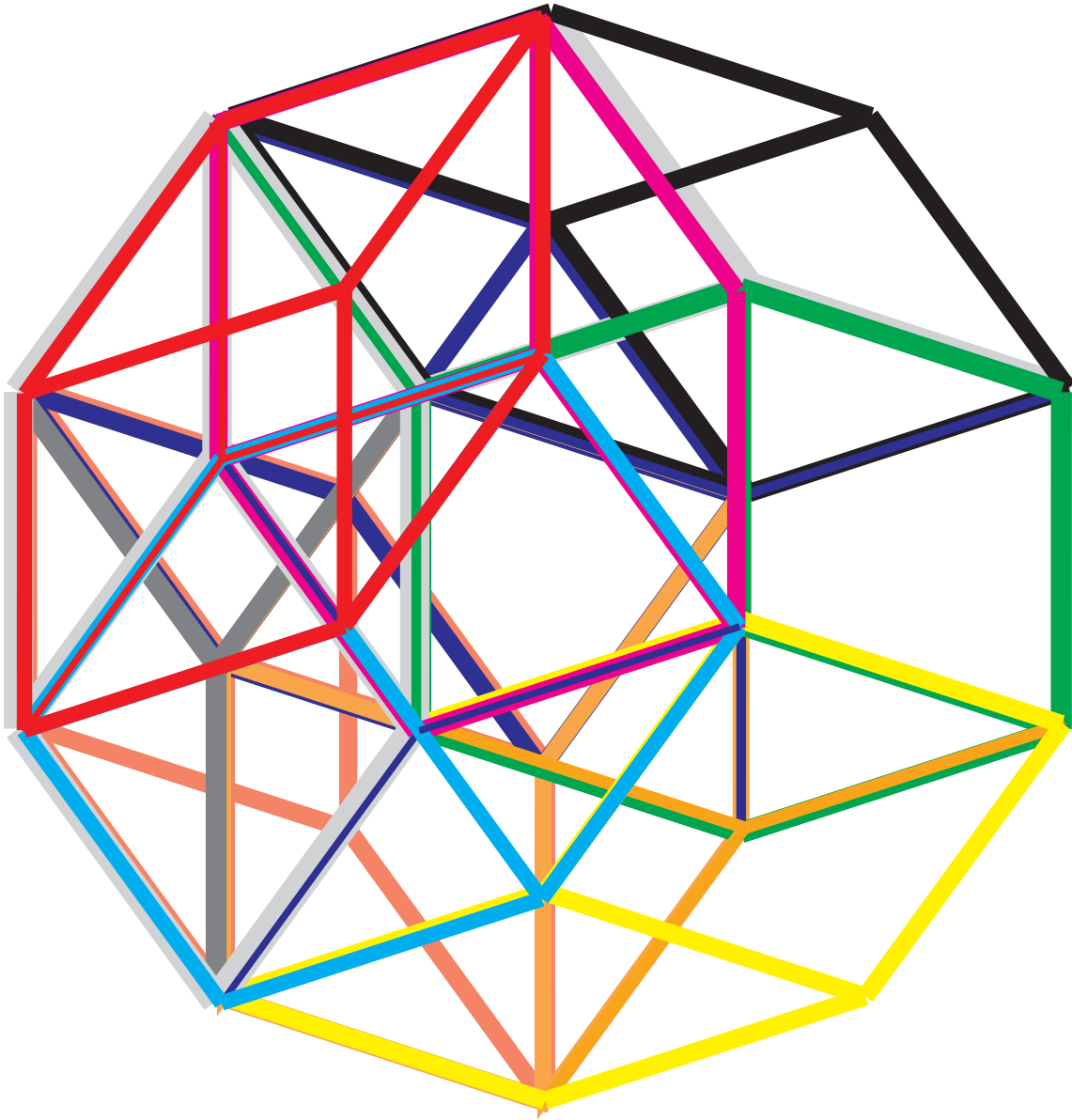


Figure 37: Front 10 rhombi give  $\sigma_2[5]$ ; back 10 rhombi give  $\tau_2[5]$ . The ball consists of 10 3-cubes.

